最少原理의 적용에 의한 二重熱容量系의 最適斷續溫度制御方式에 관한 연구

논 문
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Pontryagin's Minimum Principle Applied to a Double Capacitive Thermal System

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Abstract

This study intends to investigate the optimal switching modes of a double-capacitive thermal system under different constraints on the state and the control variable, by the application of the Pontryagin's Minimum Principle. Throughout the development, the control effort is assumed to have two modes of state; M or zero and the terminal times being fixed.

In the first part of this study, the Principle is discussed under various conditions for this particular problem, with different criterion functions and in the same time imposing a certain constraints; i) on the terminal states, ii) on functions of the terminal states. Depending upon the upper bound value of the control vector, possible driving modes of the states are studied from which particular optimal driving modes are extracted so as to meet the specified constraints and boundary conditions imposed in the problem.

Numerical solutions are evaluated for an over-damped, double-capacitive thermal plant and the optimal solutions: the switching mode, the optimal switching time, and the control effort are compared with the analytical results, in the second part of this work, to confirm the development.

1. INTRODUCTION

In general, when the dynamic equation of the system and the performance in dexare linear in the control variables, the direct application of the Minimum Principle yields the switching function in an implicit function of $u(t)$ and the usual procedure of selecting the optimal control effort so as to minimixe the Hamiltonian function fails to provide reasonable usuful information about the optimality. Particulally, when the switching junction becomes identically zero over some finite time interval and the Hamiltonian function ceases to be an explicit function of $u(t)$, the usual application of the Principle does not offer any information about the optimal conditions. This type of singular problems are well investigated by others. Bryson and Ho, Johnson and Gibson, and Tait offer excellent discussions on these problems.

Considering the current academic interests on the LOP problem, a study of optimal switching modes to a second-order linear thermal plant is proposed to investigate how the complete information for the optimality could be attained, with possibly less effort in the calculational hinderences of solving the two point boundary value problems.

2. BASIC THEORY

Let the dynamic equation of the system given in the following vector differential equation;

$$\dot{x}(t)=f[x(t),u(t),t]=F\cdot x(t)+G\cdot u(t) \quad (1)$$

with $x(t_o)$ given and the terminal times t_o, and t_f are both specified. Following inequality and equality constraints are imposed on the control vector $u(t)$ and on functions of the terminal states;

$$u(t)-M\leq 0 \quad (2)$$

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and

$$\psi[x(t_f), t_f]=0 \tag{3}$$

The upper limit of $u(t)$, M can be free or fixed, depending whether $x(t_f)$ is specified to have prescribed values or not. It can be well expected that the nominal switching solution, if it exists, will always require the control variables to be one point or another of the boundary in the feasible control region. Let the performance index, J, be in the following form;

$$J[x(t), u(t), t]=\phi[x(t_f), t_f]+\upsilon^T\psi[x(t_f), t_f] +\int_{t_o}^{t_f}\{\lambda^T(t)\ [f_{(x(t), u(t), t)}-\dot{x}(t)] +L_{(x(t), u(t), t)}\}\,dt \tag{4}$$

where υ^T is the undetermined Lagrangian multiplier. If

$$\Phi_{(x(tf), tf]}=\phi_{(x(tf'), tf]}+\upsilon^T\psi[_{(xtf), tf}] \tag{5}$$

then, equation 4 reduces to;

$$J_{(x(t), u(t), t]}=\Phi[_{x(tf), tf}]+\int_{t_o}^{t_f}\{L_{(x(t), u(t), t)}+\lambda^T(t)\cdot [f_{(x(t), u(t), t)}-\dot{x}(t)]\}\,dt \tag{6}$$

When the terminal times at both ends are fixed as in this study, the first variation in J results;

$$\begin{aligned}\delta J&=\left[\left(\frac{\partial\Phi}{\partial x}-\lambda^T(t)\right)\cdot\delta x\right]_{t=tf}+\left[\lambda^T(t)\cdot\delta x\right]_{t=to} +\int_{t_o}^{t_f}\left[\left(\frac{\partial H}{\partial x}+\dot{\lambda}^T(t)\right)\cdot\delta x+\frac{\partial H}{\partial u}\cdot\delta u\right]dt\\ &=\left[\left(\frac{\partial\Phi}{\partial x}-\lambda^T(t)\right)\cdot\delta x\right]_{t=tf} +\int_{t_o}^{t_f}\left[\left(\frac{\partial H}{\partial x}+\dot{\lambda}^T(t)\right)\cdot\partial x+\frac{\partial H}{\partial u}\cdot\delta u\right]\cdot dt\end{aligned} \tag{7}$$

where the Hamiltonian function, H is formed as;

$$H_{x(t), u(t), t}=L_{(x(t), u(t)\,t)}+\lambda^T(t)\cdot f_{(x(t), u(t), t)} \tag{8}$$

3. FORMULATION OF THE PROPORSAL

1. CASE A: As the first case, It is required to bring the initial temperature $x(t_o)$ to the same value at the fixed terminal time $t=t_f$ but the final value of temperature is not specified to have a certain prescribed value.

Let $\dot{\lambda}^T$ be in the following form;

$$\dot{\lambda}^T(t)=-\frac{\partial H}{\partial x}=-\frac{\partial L}{\partial x}-\lambda^T(t)\frac{\partial f}{\partial x} =-\frac{\partial L}{\partial x}-\lambda^T(t)\cdot F \tag{9}$$

with the boundary conditions;

$$\lambda^T(t_f)=\frac{\partial\Phi}{\partial x(t_f)}=\frac{\partial\phi}{\partial x(t_f)}+\upsilon^T\frac{\partial\psi}{\partial x(t_f)} \tag{10}$$

then, for the optimality,

$$\delta J=\int_{t_o}^{t_f}\frac{\partial H}{\partial u}\cdot\delta u\cdot dt=\int_{t_o}^{t_f}H_u\cdot\delta u\cdot dt\geq 0 \tag{11}$$

along $\quad u(t)-M\leq 0$

For this particular case, function of the terminal state is constrained to have the same temperature whose value is not specified, and hence, the performance index can be chosen to be in the following form, by letting $L=\phi=O$ and $\Phi=\upsilon^T\psi[_{x(tf), tf}]$;

$$J[_{x(t), u(t), t}]=\upsilon^T\psi[_{x(tf), tf)}] +\int_{t_o}^{t_f}\{\lambda^T(t)[f_{(x(t), u(t), t)}-\dot{x}_{(x(t), u(t), t)}]\}\,dt$$

Hence, for the optimality, the influence function reduces to;

$$\dot{\lambda}^T(t)=-\lambda^T(t)\cdot F \tag{12}$$

with boundary conditions;

$$\lambda^T(t_f)=\upsilon^T\frac{\partial\psi}{\partial x(t_f)} \tag{13}$$

2. CASE B: As the second case, some of the final temperatures are specified to have a certain value(s) at the fixed terminal time. It can be noticed that the case is almost the same as before except the prescription of some of the terminal temperature at the end terminal time. Since the terminal times at both ends are fixed, equation 10 is no longer necessarily valid and hence some modifications are neccessary in the development thus far obtained. If it is desired to constrain some of the components of the state vector to have prescribed values at $t=t_f$, the performance index can be formed in the same expression as in equation 4 or in equation 6, with the Hamiltonian function $H=L+\lambda^T\cdot f$. The first variation in J due to the variation in control $u(t)$ from the nominal trajectory, then, should be in the same form as in equation 7, that is;

$$\delta J=\left[\left(\frac{\partial\Phi}{\partial x}-\lambda^T\right)\delta x\right]_{t=tf}+\int_{t_o}^{t_f}\left[\left(\frac{\partial H}{\partial x}+\dot{\lambda}^T\right)\delta x +\frac{\partial H}{\partial u}\cdot\delta u\right]dt$$

If x_i, the i-th component of the vector $x(t)$ is prescribed at $t=t_f$, then, it follows that the admissible variation must produce $\delta x_i(t_f)=0$ and it is not necessary to hold $(\partial\Phi/\partial x_i-\lambda_i)=0$ at $t=t_f$ but it can be rather arbitrary value.

Here, another Lagrangian multiplier $T\mu(t)$ is introduced and the criterian function I is formed in the following form;

$$I=\mu^T\phi_{(x(t_f),\ t_f)}+\upsilon^T\psi_{(x(t_f),\ t_f)}+\int_{t_0}^{t_f}L_{(x(t),\ u(t),\ t)}\cdot dt \tag{14}$$

then, with the same arguments in the proceeding case, the performance index reduces into the more generalized form, that is;

$$J_{(x(t),\ u(t),\ t)}=\mu^T\phi_{(x(t_f),\ t_f)}+\upsilon^T\psi_{(x(t_f),\ t_f)}+\int_{t_0}^{t_f}\{\lambda^T(t)[f_{(x(t),\ u(t),\ t}-\dot{x}_{(t)}]+L_{x(t),\ u(t),\ t)}\}dt \tag{15}$$

let

$$\Phi[_{x(t_f),\ t_f)}=\mu^T\phi_{(x(t_f),\ t_f)}+\upsilon^T\psi_{(x(t_f),\ t_f)}] \tag{16}$$

then, equation 15 reduces to the same form as in equation 6.

As in the previous case, for the optimality, it is desired to select the influence function to be in the following form;

$$\dot{\lambda}^T(t)=-\frac{\partial H}{\partial x}=-\frac{\partial L}{\partial x}-\lambda^T(t)\cdot F \tag{17}$$

with the boundary conditions;

$$\lambda^T{}_{j(t_f)}=\frac{\partial\Phi}{\partial x_j(t_f)}=\begin{cases}\mu_j & j=1,2,\cdots,q\\ \upsilon^T_k\dfrac{\partial\psi_k}{\partial x(t_f)} & j=q+1,\cdots,n\\ & k=1,2,\cdots,r\end{cases} \tag{18}$$

where r equals to the number of the constraining functions on the terminal states at $t=t_f$ and $\mu_j=0$ for $q+1,\ q+2,\cdots\cdots n$.

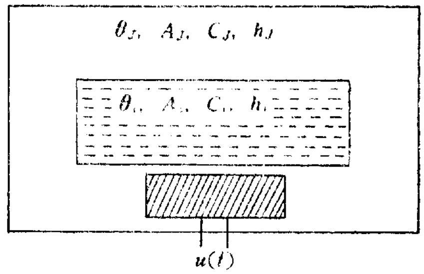
4. STATE MODEL OF THE PROPOSED THERMAL SYSTEM

A double-capacitive thermal system depicted in figure 1-(a) is proposed to be temperature controlled by manipulation of the step type heat input $u(t)$. The necessary simplifying assumptions are i) the fluid inside the tank are at a uniform temperature and ii) time delay in temperature distribution are assumed to be negligible.

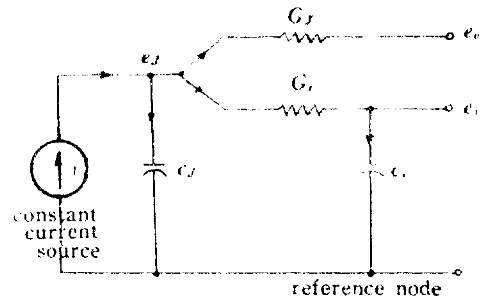
From the equivalent relation between the thermal and electrical system shown in Figure 1-(b), the topological graph is drawn for the proposed thermal system and a tree is selected as shown in Figure 1-(c), from which the following equivalent system equations are obtained;

$$G_i(e_J-e_i)=C_i(de_i/dt)$$
$$G_J(e_J-e_o)+G_i(e_J-e_i)+C_J(de_J/dt)=i(t) \tag{19}$$

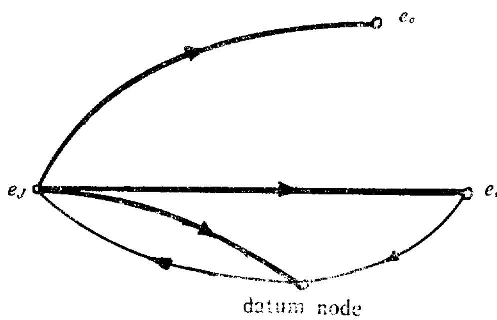
or substituting their corresponding thermal equiva-



(a) thermal system proposed



(b) electrical equivalence of the thermal system



(c) topological graph of the system and a tree selected for the system equation

Fig. 1. Temperature controlled double-capacitive thermal system proposed for the study.

lent constants and variables;

$$C_J\cdot\dot{\theta}_J=A_o\cdot h_o(\theta_o-\theta_J)+A_i\cdot h_i(\theta_i-\theta_J)+u(t)$$
$$C_i\cdot\dot{\theta}_i=A_i\cdot h_i(\theta_J-\theta_i) \tag{20}$$

When the state vector is selected with respect to the datum v variable, the outer side temperature of the tank, the expression of the system model reduces further to the following matrix equation;

$$\dot{x}(t)=\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-(A_oh_o+A_ih_i)/C_J & A_ih_i/C_J\\ A_ih_i/C_i & -A_ih_i/C_i\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}1/C_J\\ O\end{pmatrix}u(t) \tag{21}$$

5. DERIVATION OF THE OPTIMALITY CONDITIONS

Case A: When this case is treated as having a constraining function on states at the end terminal time, then the following function, can be formulated assuming the undetermined constant υ be positive value;

$$\phi_{(x(tf),\,tf)}=0 \text{ and } \psi_{(x(tf),\,tf)}=\pm\upsilon[x_1(t_f)-x_2(t_f)] \quad (22)$$

Hence,

$$\Phi_{(x(tf),\,tf}=\psi_{(x(tf),\,tf)}=\pm\upsilon[x_1(t_f)-x_2(t_f)] \quad (23)$$

The influence function becomes;

$$\dot{\lambda}(t)=\begin{pmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{pmatrix}=\begin{pmatrix}(A_oh_o+A_ih_i)/C_J & -A_ih_i/C_i\\ -A_ih_i/C_J & A_ih_i/C_i\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \quad (24)$$

With the boundary conditions;

$$\lambda^T(t_f)=\pm\upsilon[1-1] \quad (25)$$

The optimal control law gives;

$$u(t)=M \quad \text{when} \quad \lambda_1<0$$
$$u(t)=O \quad \text{when} \quad \lambda_1>0 \quad (26)$$

Equation 21 and equation 24 together with the split-boundary conditions; $x_{(to)}$ and $\lambda_{(tf)}$ gives the optimality condition and from equation 26, the optimal control mode can be depicted.

Case B: When $x_l(t_f)$ is prescribed to have temperature K_l at the end terminal time and x_2 (t_f) is left free, the criterian function I can be expressible in the following form, by letting $\psi=\Phi=L=O$,

$$I=\phi=\pm\mu_l[x_l(t_f)-K_l] \quad (27)$$

then, it results the same influence function as shown by equation 24 with the change in the boundary conditions, that is;

$$\lambda(t_f)=\begin{pmatrix}\pm\mu_l\\ O\end{pmatrix} \quad (28)$$

On the contrary, if $x_2(t_f)$ is specified to have a certain temperature K_2 at the terminal time and $x_l(t_f)$ is left open at this time,

$$I=\phi=\pm\mu_2[x_2(t_f)-K_2] \quad (29)$$

Also, in this case, the similar change is noticed in the boundary conditions;

$$\lambda(t_f)=\begin{pmatrix}O\\ \pm\mu_2\end{pmatrix} \quad (30)$$

To obtain the full information for the optimal control, two matrix equations; the dynamic equation of the system and the influence equation sould be solved simultaneously with the boundary conditions given by equation 25, equation 28 or equation 30 thus far obtained under the specific conditions for each corresponding cases. The terminal conditions on the states go into the side conditions to determine the undetermined Lagrangian multipliers; μ's and υ's. In either case, it results the same control law described by equation 26.

6. OPTIMAL SWITCHING STRATEGIES

Various computational techniques are available to solve the vector differential equations simultaneously, with the split-boundary conditions and side conditions. When the system is over-damped and the coefficient matrix of the system have distinct eigenvalues, the general solution of the influence equation can be expressible as in the following form;

$$\lambda(t)=\begin{pmatrix}A & B\\ A' & B'\end{pmatrix}\begin{pmatrix}e^{-p_1t}\\ e^{-p_2t}\end{pmatrix} \quad (31)$$

where A, B, A', and B' should be determined so as to saticify the boundary conditions specified in each of the previous case. p_1 and p_2 are the distinct igenvalues of the matrix F. Since the previous development shows that the optimal control law is depend upon $\lambda_1(t)$ function only, possible optimal

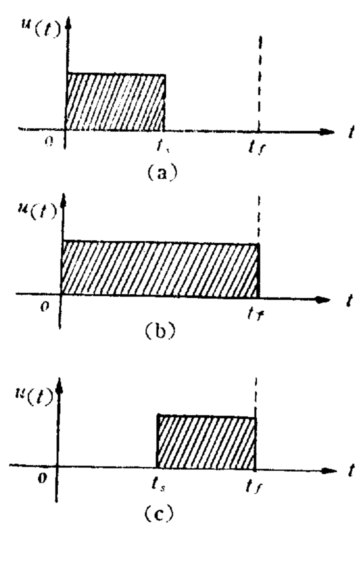
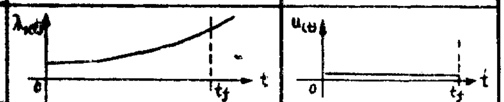
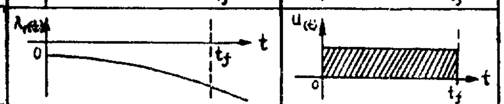
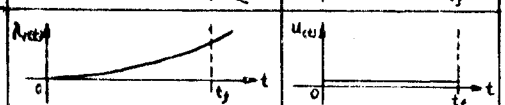
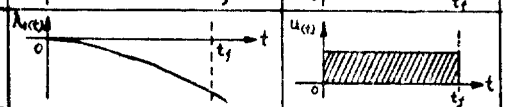
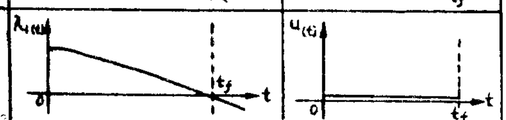
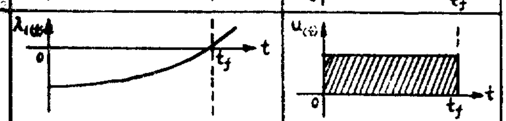
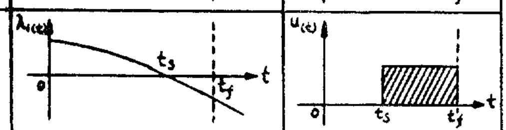
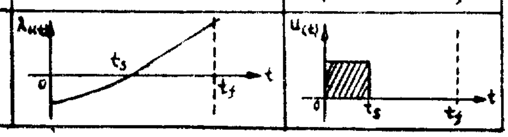


Fig. 2. Possible effective control modes.

Table 1. Possible optimal control modes

mode	conditions on A & B	$\lambda_1(0)$	$\lambda_1(0)$	variation of 1(t)	control modes
1	$A \geq 0,\ B > 0$ or $A > 0,\ B \geq 0$ or $\lvert A\rvert < \lvert B\rvert,\ A < 0,\ B > 0$	+	+		
2	$A \leq 0,\ B < 0$ or A 0, B 0 or A B, A 0, B 0	−	−		
3	$\lvert A\rvert = \lvert B\rvert,\ A < 0,\ B > 0$	0	+		
4	$\lvert A\rvert = \lvert B\rvert,\ A > 0,\ B < 0$	0	−		
5	$A/B = -k\exp(-p_1 - p_2)t_f$ $A > 0$ where $p_1 > p_2$ $\&\ k > 0$	+	0		
6	$A/B = -k\exp(-p_1 - p_2)t_f$ $A < 0$ where $p_1 > p_2$ $\&\ k > 0$	−	0		
7	$\lvert A\rvert > \lvert B\rvert,\ A > 0,\ B < 0$	+	−		
8	$\lvert A\rvert > \lvert B\rvert,\ A < 0,\ B > 0$	−	+		

control modes are evaluated for each of the corresponding boundary conditions and they are tabulated in Table 1.

In Table 1, mode 1, mode 3 and mode 5 are the stationary case of the system and they can be discarded from our interest. case 2, case 4 and case 6 reveal the same mode of control and the effective control modes can be summerized as in Figure 2.

7. ANALYSIS OF THE DEVELOPED RESULTS

To take the numerical illustration of the previous development, let $A_i h_i = A_o h_o = 0.1$, $C_i C_J = 1.0$ and $t_f = 20$ sec so that the system under study is over-damped. With these numerical constants, it can be shown that both of the states are controllable by the single scalar input $u(t)$.

1. Case A: The switching function $\lambda_1(t)$ for this case is found to be;

$$\lambda_1(t) = \frac{\pm\upsilon}{2.24}\left[2.62e^{0.262(t-t_f)} - 0.38e^{0.038(t-t_f)}\right]$$

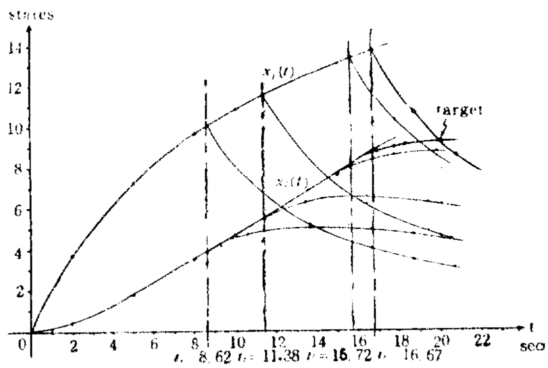
with $\lambda_1(0) = \mp 0.07313\,\upsilon$ and $\lambda_1(t_f) = \pm\upsilon$

The variation of the switching function reveals this case can posses two possible control modes, namely mode 7 and mode 8 in table 1. The optimal control law gives $u(t) = M$ when $\lambda_1(t)$ is less than zero or equivalently;

$$u(t) = M \quad \text{for} \quad 0 \leq t \leq t_s$$
$$u(t) = 0 \quad \text{for} \quad t_s \leq t \leq t_f$$

Depending on whether the final temperature is specified to have a certain value or not, this control case can be splitted into the following two different cases.

Case A-1; When $x_1(t_f) = x_2(t_f) = K$ is specified.



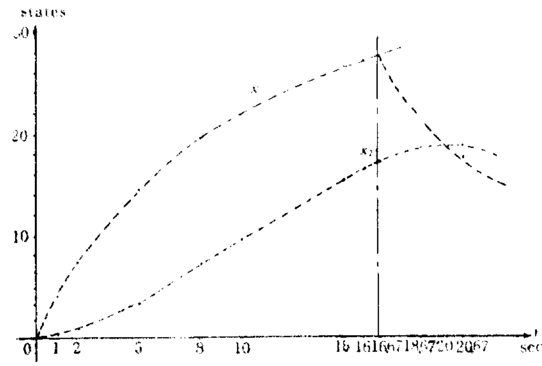


Fig. 3-(1)   **Fig. 3-(2)**

Fig. 3. Transition sequence for the optimal and off-optimal switching for case A.

Table 2. State sequences for the optimal driving of the state for case A.

time in sec	state $x_1(t)$ and $x_2(t)$ for heating up				state $x_1(t)$ and $x_2(t)$ for t_s, t_f			
	case A-1		case A-2		case A-1		case A-2	
0.00	0.00	0.00	0.00	0.00				
1.00	2.07	0.18	4.14	0.36				
2.00	3.79	0.46	7.58	0.92				
5.00	7.33	1.76	14.66	3.52				
8.00	9.73	3.61	19.46	7.22				
8.62	10.07	3.91	—	—				
10.00	10.87	4.76	21.74	9.52				
11.38	11.55	5.55	—	—				
15.00	13.12	7.66	26.24	15.32				
15.72	13.39	8.05	—	—				
16.00	13.48	8.18	26.96	16.36				
16.67	13.70	8.50	27.39	16.99	13.696	8.496	27.39	16.99
17.12	14.03	9.02	—	—	—	—	—	—
17.67	—	—	—	—	12.001	8.896	—	—
18.67	—	—	—	—	10.634	9.128	21.27	18.25
19.67	—	—	—	—	9.544	9.207	—	—
20.67	—	—	—	—	8.660	9.225	17.32	18.45

for case A-1, $x_1(t_f)=x_2(t_f)=9.214$ and for case A-2, M=4.48

In effect, in this case the problem can be stated as a fixed terminal case at both ends and it is required to find the optimal switching time, t_s and the upper limit value of u(t) which meets the constraints and the boundary conditions.

From the obtained control law, $u(t)=0$ for$[t_s, t_f]$. Hence, the state sequence during this time interval becomes;

$$x(t)=\Phi(t, t_s), x(t_s)$$

and at $t=t_f$,

$$x(t_f)=\Phi(t_f, t_s)\cdot x(t_s) \tag{32}$$

where $\Phi(t, \tau)$ is the transition matrix of the thermal system.

Solving the above equation for $x(t_s)$,

$$x(t_s)=\Phi^{-1}(t_f, t_s)\cdot x(t_f) \tag{33}$$

The state sequence for the time interval $[O, t_s]$ can be obtained from the following relation;

$$x(t)=\Phi(t, to)\cdot x(to)+\int_{to}^{t}\Phi(t, \tau)\cdot G\cdot u^*(\tau)\cdot d\tau \tag{34}$$

for $to\leq t\leq t_s$

from which the state at the switching time results;

$$x(t_s)=\Phi(t_s, t_o)\cdot x(t_o)+\int_{to}^{t_s}\Phi(t_s, \tau)\cdot G\cdot u^*(\tau)\cdot d\tau \tag{35}$$

Equating the right side of equation 33 and equation 35 and solving for $x(t_f)$ to yield;

$$x(t_f)=\Phi^{-1}(t_s,t_f)\cdot\Phi(t_s,t_o)\cdot x(t_o) + \Phi^{-1}(t_s,t_f)\int_{to}^{t_s}\Phi(t_s,\tau)\cdot G\cdot u^*(\tau)\cdot d\tau$$
$$=\Phi(t_f,t_o)\cdot x(t_o)+\int_{to}^{t_s}\Phi(t_f,\tau)\cdot G\cdot u^*(\tau)\cdot d\tau \quad (36)$$

For $x(t_o)=t_o=0$,

$$x(t_f)=M\int_o^{t_s}\Phi(t_f,\tau)\cdot G\cdot d\tau=M\int_o^{t_s}\theta^T(\tau,t_f)\cdot G\cdot d\tau \quad (37)$$

where $\theta^T(\tau,t_f)$ is the transition matrix of the adjoint system and $u^*(t)$ is the optimal control effort.

Equation 37 now can be solved for the unknown, t_s, to obtain the switching time. Once the optimal switching time is found, the complete transition sequences of the state for whole time interval can be found from the following expression;

$$x(t)=\Phi(t,t_o)\cdot x(t_o)+\int_{to}^{t_s}\Phi(t,\tau)\cdot G\cdot u^*(\tau)\cdot d\tau \quad (38)$$
$$\text{for } t_o\leq t\leq t_s$$

and

$$x(t)=\Phi(t,t_s)\cdot x(t_s)+\int_{ts}^{t_f}\Phi(t,\tau)\cdot G\cdot u^*(\tau)\cdot d\tau \quad (39)$$
$$\text{for } t_s\leq t\leq t_f$$

Numerical results for the optimal control are tabulated in Table 2 and the transition sequences of the state are plotted in Figure 3-(1). To show the early or late switching sequences from the optimal switching time, four different values of t_s are taken and the off-optimal transition sequences are imposed on the same sheet for the comparison porpose with the optimal driving sequence.

Case A-2: When $x_1(t_f)$ and $x_2(t_f)$ are not specified but it is required to reach the final states being the equal temperature.

This is almost the same problem as the previous case except the final states are not specified. The optimal switching time can be optained by the same development as in the previous case. The terminal temperature at $t=t_f$ shall be obtained from one of the state equation by the substitution of the switching time obtained. In this case, the upper bound value of the control signal should be specified. The transition of the state for this case are lotted in Figure 3-(2) bye the dashed lines.

2. Case B: As the second case, one of the terminal temperature is specified to have a certain prescribed value at the fixed terminal time $t=t_f$ but the other state temperature is left free.

Case B-1; When $x_1(t_f)$ is prescribed to have temperature K_1 at the end terminal time and $x_2(t_f)$ is left open.

The switching function results;

$$\lambda_1(t)=\frac{\pm\upsilon}{2.24}[1.62e^{0\cdot262(t-t_f)}+0.62e^{0\cdot038(t-t_f)}]$$

mode 1 in Table 1 is the effective optimal control mode. In effect, this results show a contineous driving problem of finding an adequate value of M, using the dynamic equation of the system and no switching action is involved in this driving mode. Let this value of $u(t)$ be denoted by M_c. From the dynamic equation of the system, the state transition for the time interval $[o,\ t_f]$ is;

$$x(t)=\Phi(t,t_o)\cdot x(t_o)+\int_{to}^{t}\Phi(t,\tau)\cdot G\cdot M_c\cdot d\tau \quad (40)$$

with $x(t_o)=0$ and $t_o=0$,

$$x(t)=\int_o^t\Phi(t,\tau)\cdot G\cdot M_c\cdot d\tau \quad (41)$$

The numerical sequences of the transition of states are evaluated and plotted in Figure 5-(1). Obviously, the minimum value of $u(t)$ which can transfer the state along this path is given from the expression below;

$$M_c=2.24x_1(t_f)/(22.4-6.2e^{-0\cdot262t_f}-16.2e^{-0\cdot038t_f})$$

When $u(t)$ is less than this critical value, then one is never able to transfer the state to the specified temperature within the fixed terminal time. On the other hand, when $u(t)$ is allowed to have larger than this M_c, then one may wonder, intuitively, that mode 7 and mode 8 in Table 1 are also possible driving modes of the state to the specified value. It is interesting to note that the proceeding development does not involve any one of this control mode. The performance index is modified, at this time, in the following form;

$$\phi=\pm\upsilon[x_1(t_f)-K_1],\quad L=\pm u,\quad \text{and } \phi=\Phi=0$$

That is, another constraint on the power consumption is imposed, then, it results the influence equation and the boundary conditions in the same expression as before. The optimal control law, now at this time, yields to;

$$\frac{\partial H}{\partial u}=\frac{\partial}{\partial u}[L+\lambda^T\cdot f]=\frac{\partial}{\partial u}\{\pm u+\lambda^T[F\cdot x+G\cdot u]\}$$

$$=\frac{\partial}{\partial u}[\pm u+\lambda^T\cdot G\cdot u]=(\pm 1+\lambda_1)<0$$

that is,

$u(t)=M$ for $\lambda_1<-1$ for minimum power consumption

$u(t)=M$ for $\lambda_1<+1$ for maximum power consumption

Figure 4 shows the variation of the switching function and the desired control effort for both cases.

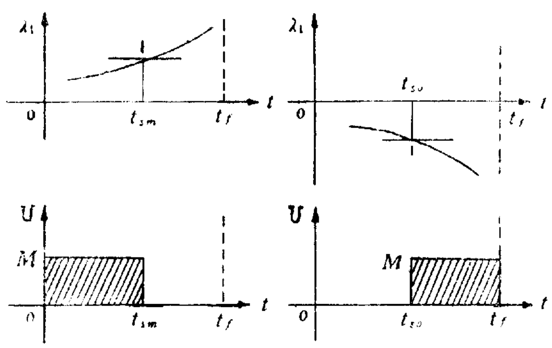


Fig. 4. Driving modes of the state for max. & min. power consumption for case B.

For the case of maximum consumption, the optimal switching time, t_s may be obtained from the following expression;

$$K/M+3.4136=0.01462556e^{0.262t_s}+3.396731e^{0.038t_s}$$

Numerical results with $x_1(t_f)=10$, $t_f=20$ sec, and with $M=5.6928$ are tabulated below;

for max. power consumption
switching time $=t_s=10.0$ sec
$u(t)=5.6928$ for $0\leq t\leq t_s$
$u(t)=0$ for $t_s\leq t\leq t_f$
power consumption $=56.928$

for min. power consumption
switching time $=t_{so}=17.8$ sec
$u(t)=0$ for $0\leq t\leq t_{so}$
$u(t)=5.6928$ for $t_{so}\leq t\leq t_f$
power consumption $=12.524$

The state sequences for each cases: the direct driving, the driving with maximum power consumption and the driving with minimum power consumption, are tabulated in Table 3 and are compared in Figure 5-(1).

Case B-2; Here, at this point, it may be reasonable to consider the reversed case of the previous discussion. At this time, temperature $x_2(t_f)$ is specified at the end terminal time to have a certain prescribed value K_2 but $x_1(t_f)$ is left free.

The switching function $\lambda_1(t)$ gives;

$$\lambda_1(t)=\frac{\pm\upsilon}{2.24}[e^{0.262(t-t_f)}+e^{0.038(t-t_f)}]$$

The application of the Minimum Principle reveals that the optimal driving mode of the state is the mode 6 in Table 1. The critical value of $u(t)$ for this control mode is found from the second expression in the dynamic equation of the system. With $x_2(t_f)=K_2=10$, M_c is found to have 2.203.

As in the previous case, when M_c is allowded to have more larger value than this, then by the similar development, the switching time, t_{sm} for the maximum power consumption and the optimal switching time, t_{so} for the minimum power consumption can be obtained from the following two expressions;

$$K_2/M+5.4799=5.4889e^{0.038t_{sm}}-0.00902e^{0.262t_{sm}}$$

for max. power consumption

$$K_2/M=1+1.6964e^{-0.262(t_f-t_{so})}-11.6964e^{-0.038(t-t_{so})}$$

for min. power consumption

For $x_2(t_f)=K_2=10.0$, and with $M=4.48$, the optimal control mode is evaluated and are summerized below, for each cases;

for max. power consumption
optimal switching time $=t_{sm}$ $=9.35$ sec
$u(t)=4.48$ for $0\leq t\leq t_{sm}$
$u(t)=0$ for $t_{sm}\leq t\leq t_f$
power consumption $=41.888$

for min. power consumption
optimal switching time $=t_{so}$ $=10.65$ sec
$u(t)=0$ for $0\leq t\leq t_{so}$
$u(t)=4.48$ for $t_{so}\leq t\leq t_f$
power consumption $=41.888$

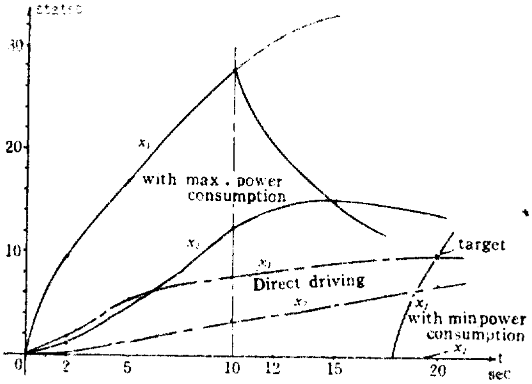


Fig. 5-(1)

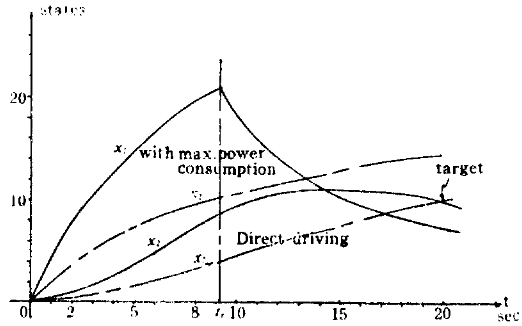


Fig. 5-(2)

Fig 5. State transitions for case B.

Table 3. State sequences for the optimal driving of the state for case B.

TIME sec	state $x_1(t)$ and $x_2(t)$ for case B-1						state $x_1(t)$ and $x_2(t)$ for case B-2					
	direct		max. power		min. power		direct		max. power		min. power	
0.00	0.000	0.000	0.000	0.000			0.000	0.000	0.00	0.00		
2.00	1.767	0.001	9.466	0.901			3.663	0.349	7.58	0.92		
5.00	4.956	1.189	18.630	4.470			7.210	1.730	14.66	3.52		
8.00	—	—	—	—			—	—	19.46	7.22		
10.00	7.349	3.218	27.637	12.096			10.690	4.680	—	—		
10.65	—	—	—	—			—	—	—	—	0.00	0.00
11.35	—	—	—	—			—	—	15.70	10.48	—	—
12.65	—	—	—	—			—	—	—	—	7.58	0.92
14.35	—	—	—	—			—	—	11.11	11.14	—	—
15.00	—	—	14.727	15.012			12.900	7.531	—	—	—	—
17.65	—	—	—	—			—	—	—	—	14.66	3.52
17.80	—	—	—	—	0.00	0.00	—	—	—	—	—	—
18.18	—	—	—	—	6.051	0.052	—	—	—	—	—	—
19.35	—	—	—	—	—	—	—	—	7.49	10.26	—	—
19.80	—	—	—	—	9.466	0.901	—	—	—	—	—	—
20.00	10.000	6.874	9.986	13.765	—	—	14.547	10.00	—	—	—	—
20.35	—	—	—	—	—	—	—	—	—	—	15.70	10.48

Once the switching time is found, the evaluation of the state sequences for each of the three cases are straightfoward and the results are plotted in Figure 5-(2). It is worthwhile to note that the power consumption for the three cases shows rather different features from those of case B-1.

8. CONCLUSIONS AND DISCUSSIONS

1. As expected, the switching function for the LOP problem proposed reveals little information about the optimal switching time and the control effort required, without virtually solving the dynamic equation of the system and the influence function. It is understood that the general hinderances for obtaining numerical solutions of the matrix differential equation with split-boundary conditions are of guessing the starting trial values for the solutions. Even though the switching function thus far obtained does not provide an explicit expression about the switching time, it was illustrated that one can, at least, depict the possible optimal control modes from the switching function and hence this provides quite reasonable means of guessing the starting point for the solutions.

2. Depending upon the upper bound value of $u(t)$, the application of the Principle gives different possibilities for the optimal solution to the problem. For case A, the control mode (a) in Figure 2 is the only optimal control mode applicable to the case when the terminal temperature is required to be the same value at $t=t_f$. The problem is, then, left for finding the switching time, t_s and the upper bound of the optimal control effort. The optimal switching time was found to be unique.

3. For case B, when $x_1(t_f)$ is apecified and $x_2(t_f)$ is left free, it has shown that three of the driving modes in Figure 2 are found to be the possible optimal control mode. Depending upon the upper limit of $u(t)$, different type of driving should be employed, that is;

i) when $M=M_c$, then the control mode (b) in Figure 2 is effective

ii) when $M\fallingdotseq M_c$, the control mode (a) or (c) in Frigue 2 is effective. It is worthwhile to note the power consumption for each of the three possible driving cases of the state. The control mode (c) showed the minimum power consumption, meanwhile, the mode (a) consumed the maximum power for achieving the desired driving of the state. For the case when $M=M_c$, $u(t)$ remains on the same state for whole time interval (o, t_f) and no switching action involved in this driving mode. This mode required the least value of $u(t)$ for reaching the same terminal states, comparing with the other cases.

4. When $x_2(t_f)$ is specified to have a certain temperature at $t=t_f$ instead of $x_1(t_f)$ and $x_1(t_f)$ is left open, M_c is found to be larger than the value obtained in case B-1 for the same final temperature. With $M\fallingdotseq M_c$ and with the control mode (b), the power consumption for (o, t_f) is found to be 44.06 which is larger than that of case B-1.

REFERENCES

1. C. D. Johnson and J. E. Gibson, "Singular Solutions in Problems of Optimal Control", IEEE Trans. on Automatic Control, pp 4-15, Jan. 1963
2. K. Tait, "Singular Problems in Optimal Control" Ph. D Thesis, Harvard Univ., 1965.
3. C. Johnson, "Singular Problems in Optimal Control", Advances in Control Systems, pp 207, Vol. 2, Academic Press, New York, 1965.
4. Ian McCausland, "Introduction to Optimal Control", John Wiley & Sons, Inc., New York, 1969.
5. M Athans & P. L. Falb, "Optimal Control", McGraw-Hill, New York, 1966.
6. R. C. Dorf, "Modern Control Systems", Addison Wesley Publication Co., 1967.
7. S. I. Gass, "Linear Programming", McGraw-Hill, New York, 1969.
8. A. E. Bryson, Jr and Y. C. Ho, "Applied Optimal Control", Blaisdell, Waltham, Massachusetts 1969.
9. O. I. Elgerd, "Control Systems Theory", McGraw-Hill, New York, 1967.
10. V. W. Eveleigh, "Adaptive Control and Optimization Ttechniques", McGraw-Hill, New York, 1967.
11. J. T. Tou, "Modern Control Theory", McGraw-Hill, New York, 1964.
12. K. Ogata, "State Space Analgsis of Control Systems", Penrtice-Hall, Englewood Cliffs, N. J., 1967.