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ON (f, g, e, u, v, λ)-STRUCTURE

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§0. Introduction.

Yano and Okumura [7] have defined (f, g, u, v, λ)-structure on an even dimensional differentiable manifold. We see that hypersurfaces of an almost contact metric manifold and submanifolds of codimension 2 of an almost Hermitian manifold admit an (f, g, u, v, λ)-structure.

Yano, Ishihara, Okumura and Yamaguchi have studied submanifolds of codimension 2 of an almost contact manifold. These submanifolds admit another structure. we call such a structure an $(f, E, U, V, e, u, v, \lambda)$-structure. If the ambient space is an almost contact metric manifold, the submanifolds admit what we call an (f, g, e, u, v, λ)-structure.

The main purpose of the present paper is to study the (f, g, e, u, v, λ)-structure and to give some properties valid in manifold with normal (f, g, e, u, v, λ)-structure.

In §1, we define and discuss $(f, E, U, V, e, u, v, \lambda)$-structure and (f, g, e, u, v, λ)-structure.

In §2, we study submanifolds of codimension 2 of an almost contact Riemannian manifold as example of the manifold with (f, g, e, u, v, λ)-structure.

In §3, we study submanifolds of codimension 3 of an almost Hermitian manifold as another example of the manifold with (f, g, e, u, v, λ)-structure.

In §4, we prove that a manifold with normal (f, g, e, u, v, λ)-structure satisfying $de=w$ is a normal contact Riemannian manifold or a manifold with f-structure.

In §5, we find some properties valid in manifold with normal (f, g, e, u, v, λ)-structure statisfying $du=w$ and $dv=\phi w$, and we prove that the vectors U and V define infinitesimal conformal transformations under certain conditions.

The last §6 is devoted to prove two theorems which characterize odd dimensional sphere.

§1. (f, g, e, u, v, λ)-structure.

Let M be an m-dimensional differentiable manifold of class C^∞, we assume that there exist on M a tensor field f of type $(1,1)$, vector fields E, U and V, 1-forms e, u and v, and a function λ satisfying the conditions:

$$f^2X=-X+e(X)E+u(X)U+v(X)V \tag{1.1}$$

for any vector field X,

$$e\circ f=0, \qquad fE=0, \tag{1.2}$$

$$u\circ f=\lambda v, \quad fU=-\lambda V, \tag{1.3}$$

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(1.4) $$v\circ f=-\lambda u, \quad fV=\lambda U,$$

and

(1.5) $$e(E)=1, \qquad e(U)=0, \qquad e(V)=0,$$

(1.6) $$u(E)=0, \qquad u(U)=1-\lambda^2, \qquad u(V)=0,$$

(1.7) $$v(E)=0, \qquad v(U)=0, \qquad v(V)=1-\lambda^2.$$

In this case, we say that the manifold has an $(f, E, U, V, e, u, v, \lambda)$-structure. Example of the manifold with $(f, E, U, V, e, u, v, \lambda)$-structure will be given in §2 and §3.

Moreover, in M with $(f, E, U, V, e, u, v, \lambda)$-structure, if there exists a positive definite Riemannian metric g such that

(1.8) $$g(X, E)=e(X), \qquad g(X, U)=u(X), \qquad g(X, V)=v(X),$$

and

(1.9) $$g(fX, fY)=g(X, Y)-e(X)e(Y)-u(X)u(Y)-v(X)v(Y)$$

for any vector fields X and Y of M, then we call such a structure a metric $(f, E, U, V, e, u, v, \lambda)$-structure ane denote it sometimes by (f, g, e, u, v, λ)-structure.

First of all, we prove

LEMMA 1.1. *Let $M'=\{P|\lambda(P)\neq 1\}$. Then the three vector fields E, U and V are linearly independent on M'.*

Proof. Since $\lambda\neq 1$, from (1.5), (1.6) and (1.7), we see that the vector fields E, U and V are non-zero. If there are three numbers a, b and c such that

$$aE+bU+cV=0,$$

then evaluating e, u and v at the above equation, we have respectively

$$e(aE+bU+cV)=a=0,$$
$$u(aE+bU+cV)=b(1-\lambda^2)=0,$$

and

$$v(aE+bU+cV)=c(1-\lambda^2)=0,$$

from which we have $a=b=c=0$. Thus E, U and V are lineaely independent.

LEMMA 1.2. *Let $M_0=\{P|\lambda(P)=0\}$. Then $(f, E, U, V, e, u, v, \lambda)$-structure is an f-structure on M_0.*

Proof. Since $\lambda=0$, from (1.5), (1.6) and (1.7), we have

(1.10) $$fE=0, \qquad fU=0, \qquad fV=0,$$

from which and Lemma 1.1, we see that the rank of matrix (f) is $m-3$.

Operating f to (1.1) and using (1.10), we get

$$f^3X+fX=0$$

for any vector field X on M_0. Thus f define an f-structure of rank $m-3$ on M_0, which is defined by Yano [5].

PROPOSITION 1.3. *A differentiable manifold with $(f, E, U, V, e, u, v, \lambda)$-structure is of odd dimension.*

Proof. If the vector fields E, U and V are linearly independent, then there are two cases:

$$1)\ u \neq 0, \quad v \neq 0, \qquad 2)\ u=0, \quad v=0.$$

We consider the first case. In this case, 1-forms e, u and v are linearly independent. We see that the manifold with (f, U, V, u, v, λ)-structure is even dimensional [7]. Thus the manifold with $(f, E, U, V, e, u, v, \lambda)$-structure is odd dimensional.

We considerer the second case. From (1.5), we see that E is a non-zero vector and e is a non-zero 1-form, then from (1.1) we get

$$f^2X = -X + e(X)E,$$

for any vector X. Thus the manifold M is odd dimensional.

Next, if the vector fields E, U and V are linearly dependent, then U and V are both zero vectors. For, if there are two numbers a and b such that

$$aU + bV = 0, \quad a^2 + b^2 \neq 0,$$

then operating f to the above equation and using (1.3), (1.4), we get

$$bU - aV = 0.$$

Comparing the two equations, we have $U = V = 0$. Then the $(f, E, U, V, e, u, v, \lambda)$-structure is an almost contact structure. Thus the manifold M is odd dimensional.

PROPOSITION 1.4. *Let w be a tensor field of type $(0, 2)$ of M defined by*

$$w(X, Y) = g(fX, Y) \tag{1.11}$$

for any vector field X of M. Then w is a 2-form.

Proof. By the definition of w and (1.9), we get

$$\begin{aligned} w(fX, fY) &= g(fX, Y) - e(fX)e(Y) - u(fX)u(Y) - v(fX)v(Y) \\ &= w(X, Y) - \lambda v(X)u(Y) + \lambda u(X)v(Y). \end{aligned}$$

On the other hand, using (1.1), we have

$$\begin{aligned} w(fX, fY) &= g(f^2X, fY) = -g(X, fY) + e(X)e(fY) + u(X)u(fY) + v(X)v(fY) \\ &= -w(Y, X) + \lambda u(X)v(Y) - \lambda v(X)u(Y). \end{aligned}$$

Comparing the two equations above, we have

$$w(X, Y) = -w(Y, X). \tag{1.12}$$

DEFINITION. The structure $(f, E, U, V, e, u, v, \lambda)$ is said to be *normal* if the Nijenhuis tensor N of f satisfies

$$S(X, Y) = N(X, Y) + 2de(X, Y)E + 2du(X, Y)U + 2dv(X, Y)V = 0 \tag{1.13}$$

for any vector fields X and Y of M.

We consider a product manifold $M \times R^3$, where R^3 is a 3-dimensional Euclidean space. Then, $(f, E, U, V, e, u, v, \lambda)$-structure gives rise to an almost complex structure J on $M \times R^3$. For, if we put

$$(J) = \begin{pmatrix} f & E & U & V \\ -e & 0 & 0 & 0 \\ -u & 0 & 0 & -\lambda \\ -v & 0 & \lambda & 0 \end{pmatrix} \tag{1.14}$$

then we can easily find that $J^2 = -I$, where I is unit matrix.

If an almost complex structure J is integrable, then the Nijenhuis tenser of J vanishes identically, that is,

$$[JX, JY]-J[JX, Y]-J[X, JY]-[X, Y]=0$$

for any vector fields X and Y of $M\times R^3$. In this case, from (1.14) we can easily prove that $S(X, Y)=0$. Thus we have

PROPOSITION 1.5. *If J is integrable, then $(f, E, U, V, e, u, v, \lambda)$-structure is normal.*

§2. Example I.

In this section, we study submanifold of codimension 2 of an almost contact Riemannian manifold as example of the manifold with (f, g, e, u, v, λ)-structure.

Let $\bar{M}$ be a $(2n+1)$-dimensional differentiable manifold with an almost contact metric structure (ϕ, ξ, η, G) such that

$$\phi^2\bar{X}=-\bar{X}+\eta(\bar{X})\xi, \tag{2.1}$$

$$\phi\xi=0, \qquad \eta(\phi\bar{X})=0, \qquad \eta(\xi)=1. \tag{2.2}$$

$$\eta(\bar{X})=G(\xi, \bar{X}), \tag{2.3}$$

$$G(\phi\bar{X}, \phi\bar{Y})=G(\bar{X}, \bar{Y})-\eta(\bar{X})\eta(\bar{Y}), \tag{2.4}$$

for any vector flelds $\bar{X}$ and $\bar{Y}$ on $\bar{M}$.

If in an almost contact Riemannian manifold the tensor defined by

$$S(\bar{X}, \bar{Y})=N(\bar{X}, \bar{Y})+d\eta(\bar{X}, \bar{Y})\xi, \tag{2.5}$$

vanishes identically, the structure is said to be normal and the manifold $\bar{M}$ is called a normal contact Riemannian manifold. In a normal contact Riemannian manifold we have the following identities for any vectors $\bar{X}$ and $\bar{Y}$

$$(\bar{\nabla}_{\bar{X}}\eta)(\bar{Y})=G(\phi\bar{X}, \bar{Y}). \tag{2.6}$$

$$(\bar{\nabla}_{\bar{X}}\phi)\bar{Y}=\eta(\bar{Y})\bar{X}-G(\bar{X}, \bar{Y}), \tag{2.7}$$

where $\bar{\nabla}$ denotes the covariant differentiation with respect to G.

Let M be a $(2n-1)$-dimensional submanifold imbedded in $\bar{M}$ with imbedding map i: $M\rightarrow\bar{M}$. Let i_* be the differential of i and $\bar{X}$ the corresponding vector field for any vector field X on M, that is, $\bar{X}=i_*X$. Then the induced metric g on M is given by

$$g(X, Y)=G(\bar{X}, \bar{Y}) \tag{2.8}$$

for any vector fields X and Y on M. We assume that the vector field ξ belong to the tangent space of $i(M)$, then there exist a tangent vector field E and an 1-form e on M such that

$$\xi=\bar{E}, \qquad e(X)=\eta(\bar{X}). \tag{2.9}$$

It is easy to see [1] that we can define a tensor field f of type (1, 1), the vector fields U and V, 1-forms u and v, and scalar field λ on M by

$$\phi\bar{X}=\overline{fX}+u(X)C+v(X)D, \tag{2.10}$$

$$\phi C=-\bar{U}+\lambda D, \qquad \phi D=-\bar{V}-\lambda C, \tag{2.11}$$

where C and D are two mutually orthogonal unit vectors of $\bar{M}$ normal to $i(M)$.

Operating (2.10) with ϕ and taking account of (2.1) and (2.11), we get

(2.12) $$f^2X=-X+e(X)E+u(X)U+v(X)V,$$

(2.13) $$u(fX)=\lambda v(X), \quad v(fX)=-\lambda u(X).$$

Similarly, we have from (2.11)

(2.14) $$fU=-\lambda V, \quad fV=\lambda U.$$

(2.15) $$u(U)=v(V)=1-\lambda^2 \quad u(V)=v(U)=0.$$

From (2.2) and (2.9), we get

(2.16) $$e(E)=1, \quad e(fX)=0,$$

(2.17) $$fE=0, \quad e(U)=0, \quad e(V)=0.$$

On the other hand, substituting (2.10) into the equation (2.4), we obtain

(2.18) $$g(fX, fY)=g(X, Y)-e(X)e(Y)-u(X)u(Y)-v(X)v(Y).$$

Equations (2.12)$\sim$(2.18) show that a submanifold of codimension 2 of an almost contact Riemannian manifold admits a (f, g, e, u, v, λ)-structure.

Let ∇ be the covariant differentiation with respect to g on M. Then the equations of Gauss and Weingarten are given by

(2.19) $$\bar{\nabla}_{\bar{X}}\bar{Y}=\overline{\nabla_X Y}+h(X, Y)C+k(X, Y)D,$$

(2.20) $$\bar{\nabla}_{\bar{X}}C=-\overline{HX}+l(X)D, \quad \bar{\nabla}_{\bar{X}}D=-\overline{KX}-l(X)C,$$

where h and k are the second fundamental tensors and l the third fundamental tensor and H and K are tensors of type (1,1) such that

$$h(X, Y)=g(HX, Y), \quad k(X, Y)=g(KX, Y).$$

Now differentiating (2.10) covariantly on the submanifold M, we obtain

$$\begin{aligned}(\bar{\nabla}_{\bar{X}}\phi)\bar{Y}&+\phi[\overline{\nabla_X Y}+h(X, Y)C+k(X, Y)D]\\ &=\overline{(\nabla_X f)Y}+\overline{f(\nabla_X Y)}-u(Y)\overline{HX}-v(Y)\overline{KX}+[u(\nabla_X Y)+(\nabla_X u)(Y)+h(X, fY)\\ &\quad -v(Y)l(X)]C+[v(\nabla_X Y)+(\nabla_X v)(Y)+k(X, Y)+u(Y)l(X)]D.\end{aligned}$$

Thus, if $\bar{M}$ is a normal contact Riemannian manifold, then from (2.7) and (2.9) we have

(2.21) $$(\nabla_X f)Y=e(Y)X-g(X, Y)E-h(X, Y)U-k(X, Y)V+u(Y)HX+v(Y)KX,$$

(2.22) $$(\nabla_X u)(Y)=-\lambda k(X, Y)-h(X, fY)+v(X)l(Y),$$

(2.23) $$(\nabla_X v)(Y)=\lambda h(X, Y)-k(X, fY)-u(X)l(Y).$$

Differentiating (2.6) covariantly and using (2.9), we get

$$(\nabla_X e)(Y)+e(\nabla_X Y)=G(\phi\bar{X}, \bar{Y})+\eta[\overline{\nabla_X Y}+h(X, Y)C+k(X, Y)D]$$

from which

(2.24) $$(\nabla_X e)(Y)=w(X, Y).$$

Substituting (2.21)$\sim$(2.24) into the equation:

$$\begin{aligned}S(X, Y)=&f(\nabla_Y f)X-f(\nabla_X f)Y+(\nabla_{fX} f)Y-(\nabla_{fY} f)X+[(\nabla_X e)Y-(\nabla_Y e)X]E\\ &+[(\nabla_X u)Y-(\nabla_Y u)X]U+[(\nabla_X v)Y-(\nabla_Y v)X]V\end{aligned}$$

we find

(2.25) $$S(X, Y)=u(Y)(Hf+fH)X-u(X)(Hf+fH)Y$$

$$+v(Y)\,(Kf+fK)X-v(X)\,(Kf+fK)Y$$
$$+(v(X)l(Y)-v(Y)l(X))\,U+(u(Y)l(X)-u(X)l(Y))\,V.$$

Now, we assume that the connection of normal bundle of M is flat. Then we can choose two locally unit vector fields C and D in such a way that the 1-form l vanishes identically. In this case, if f anti-commute with H and K, that is,

(2.26) $$Hf+fH=0, \qquad Kf+fK=0,$$

then we get $S(X, Y)=0$. Thus we have

PROPOSITION 2.1. *Let M be a submanifold of codimension 2 of a normal contact Riemannian manifold whose connection induced in the normal bundle is flat. If the vector field ξ is tangent to M and f anti-commute with H and K, then M admits a normal (f, g, e, u, v, λ)-structure.*

If the submanifold M is a totally geodesic, then $H=K=0$. Hence we have

COROLLARY 2.2. *Let M be a totally geodesic submanifold of codimenision 2 of a normal contact Riemannian manifold whose connection induced in the normal bundle is flat. If the vector field ξ is tangent to M everywhere, then M admits a normal (f, g, e, u, v, λ)-structure.*

For a totally umbilical submanifold whose connection induced in the normal bundle is flat, we have

$$h(X, Y)=\alpha g(X, Y), \qquad k(X, Y)=\beta g(X, Y), \qquad l(X)=0$$

for any vector fields X and Y on M, where α and β are scalar functions. And consequently (2.22) and (2.23) become

(2.27) $$(\nabla_X u)(Y)=\alpha w(X, Y)-\lambda\beta g(X, Y),$$

and

(2.28) $$(\nabla_X v)(Y)=\beta w(X, Y)+\lambda\alpha g(X, Y),$$

respectively. From (2.27) and (2.28),
we get

(2.29) $$(\nabla_X u)(Y)+(\nabla_Y u)(X)=-2\lambda\beta g(X, Y).$$

and

(2.30) $$(\nabla_X v)(Y)+(\nabla_Y v)(X)=2\lambda\alpha g(X, Y).$$

Thus we have

PROPOSITION 2.3. *Let M be a totally umbilical submanifold of codimension 2 of a normal contact Riemannian manifold whose connection induced in the normal bundle is flat. If the vector field ξ is tangent to M and λ is an almost everywhere non-zero function, then the vector fields U and V define infinitesimal conformal transformations.*

§3. Example II

In this section, we study submanifolds of codimension 3 of an almost Hermitian manifold as example of the manifold with (f, g, e, u, v, λ)-structure.

Let M be a $(2n+2)$-dimensional almost Hermitian manifold and let (F, G) be the almost Hermitian structure such that

(3.1) $$F^2\bar{X}=-\bar{X},$$

(3.2) $$G(F\bar{X}, F\bar{Y})=G(\bar{X}, \bar{Y}),$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$.

Let M be a $(2n-1)$-dimensional submanifold imbedded in $\bar{M}$ with imbedding map $i: M\to\bar{M}$. Let i_* be the differential of i and $\bar{X}$ the corresponding vector field for any vector field X on M, that is, $\bar{X}=i_*X$. Then the induced metric g on M is given by

(3.3) $$g(X, Y)=G(\bar{X}, \bar{Y}),$$

for any vector fields X and Y on M.

We can define a tensor field f of type $(1, 1)$, the vector fields E, U and V, 1-forms e, u and v, and scalar functions λ_{ij} such that $\lambda_{ij}=-\lambda_{ji}$ $(i, j=1, 2, 3)$ on M by

(3.4) $$F\bar{X}=\overline{fX}+e(X)N_1+u(X)N_2+v(X)N_3,$$

(3.5) $$FN_1=-\bar{E}+\lambda_{12}N_2+\lambda_{13}N_3,$$

(3.6) $$FN_2=-\bar{U}-\lambda_{12}N_1+\lambda_{23}N_3,$$

(3.7) $$FN_3=-\bar{V}-\lambda_{13}N_1-\lambda_{23}N_2,$$

where N_1, N_2 and N_3 are three mutually orthogonal unit vectors of M normal to $i(M)$.

Operating (3.4) with F and taking account of (3.1), (3.5), (3.6) and (3.7), we get

(3.8) $$f^2X=-X+e(X)E+u(X)U+v(X)V,$$

(3.9) $$e(fX)=\lambda_{12}u(X)+\lambda_{13}v(X),\quad u(fX)=-\lambda_{12}e(X)+\lambda_{23}v(X),$$
$$v(fX)=-\lambda_{13}e(X)-\lambda_{23}u(X),$$

or

(3.10) $$e\circ f=\lambda_{12}u+\lambda_{13}v,\quad u\circ f=-\lambda_{12}e+\lambda_{23}v,\quad v\circ f=-\lambda_{13}e-\lambda_{23}u.$$

Operating F to (3.5), (3.6) and (3.7) respectively, we find

(3.11) $$fE=-\lambda_{12}U-\lambda_{13}V,\quad fU=\lambda_{12}E-\lambda_{23}V,\quad fV=\lambda_{13}E+\lambda_{23}U,$$

(3.12) $$e(E)=1-\lambda_{12}{}^2-\lambda_{13}{}^2,\quad e(U)=-\lambda_{13}\lambda_{23},\quad e(V)=-\lambda_{12}\lambda_{23},$$

(3.13) $$u(E)=-\lambda_{13}\lambda_{23},\quad u(U)=1-\lambda_{12}{}^2-\lambda_{23}{}^2,\quad u(V)=-\lambda_{12}\lambda_{13},$$

(3.14) $$v(E)=-\lambda_{12}\lambda_{23},\quad v(U)=-\lambda_{12}\lambda_{13},\quad v(V)=1-\lambda_{13}{}^2-\lambda_{23}{}^2,$$

If we put

(3.14) $$\lambda_{12}=\lambda_{13}=0,\quad \lambda_{23}=\lambda,$$

then we have from (3.10)-(3.14)

(3.10)′ $$e\circ f=0,\qquad u\circ f=\lambda v,\qquad v\circ f=-\lambda u,$$

(3.11)′ $$fE=0,\qquad fU=-\lambda V,\qquad fV=\lambda U,$$

(3.12)′ $$e(E)=1,\qquad e(U)=0,\qquad e(V)=0,$$

(3.13)′ $$u(E)=0,\qquad u(U)=1-\lambda^2,\qquad u(V)=0,$$

(3.14)′ $$v(E)=0,\qquad v(U)=0,\qquad v(V)=1-\lambda^2,$$

respectively. Equations (3.8) and (3.10)′-(3.14)′ shows that a submanifold of codimension 3 of an almost Hermitian manifold admits a (f, g, e, u, v, λ)-structure.

Next, let ∇ be the covariant differentiation with respect to g on M. Then the equations of Gauss and Weingarten are given by

$$\bar\nabla_{\bar X}\bar Y=\overline{\nabla_X Y}+h_1(X, Y)N_1+h_2(X, Y)N_2+h_3(X, Y)N_3, \tag{3.15}$$

$$\bar\nabla_{\bar X}N_1=-\overline{H_1X}-l_{12}(X)N_2-l_{13}(X)N_3, \tag{3.16}$$

$$\bar\nabla_{\bar X}N_2=-\overline{H_2X}+l_{12}(X)N_1-l_{23}(X)N_3, \tag{3.17}$$

$$\bar\nabla_{\bar X}N_3=-\overline{H_3X}+l_{13}(X)N_1+l_{23}(X)N_2, \tag{3.18}$$

where H_i are the second fundamental tensors corresponding to N_i respectively, and l_{ij} are the third fundamental tensors.

Differentiating $G(Y, N_i)=0$ covariantly, we get

$$G(\bar\nabla_{\bar X}Y, N_i)+G(\bar Y, \bar\nabla_{\bar X}N_i)=0.$$

Substituting (3.15)-(3.18) into the above equation, we get

$$h_i(X, Y)=g(H_iX, Y). \tag{3.19}$$

Differentiating (3.4) covariantly, we find

$$\begin{aligned}(\bar\nabla_{\bar X}F)\bar Y+F\overline{(\nabla_X Y)}&+h_1(X, Y)N_1+h_2(X, Y)N_2+h_3(X, Y)N_3\\ =\overline{(\nabla_X f)Y}+\overline{f(\nabla_X Y)}&-e(Y)\overline{H_1X}-u(Y)\overline{H_2X}-v(Y)\overline{H_3X}\\ &+(h_1(X, fY)+(\nabla_X e)Y+e(\nabla_X Y)+u(Y)l_{12}(X)+v(Y)l_{13}(X))N_1\\ &+(h_2(X, fY)+(\nabla_X u)Y+u(\nabla_X Y)-e(Y)l_{12}(X)+v(Y)l_{23}(X))N_2\\ &+(h_3(X, fY)+(\nabla_X v)Y+v(\nabla_X Y)-e(Y)l_{13}(X)-u(Y)l_{23}(X))N_3.\end{aligned}$$

If M is a Kaehlerian manifold, that is, if $\bar\nabla_{\bar X}F=0$, that we have

$$\begin{aligned}(\nabla_X f)Y=e(Y)H_1X+u(Y)H_2X+v(Y)H_3X-h_1(X, Y)E\\ -h_2(X, Y)U-h_3(X, Y)V,\end{aligned} \tag{3.20}$$

$$(\nabla_X e)Y=-h_1(X, fY)-u(Y)l_{12}(X)-v(Y)l_{13}(X), \tag{3.21}$$

$$(\nabla_X u)Y=-\lambda h_3(X, Y)-h_2(X, fY)+e(Y)l_{12}(X)-v(Y)l_{23}(X), \tag{3.22}$$

$$(\nabla_X v)Y=\lambda h_2(X, Y)-h_3(X, fY)+e(Y)l_{13}(X)+u(Y)l_{23}(X). \tag{2.23}$$

Substituting (3.20)-(3.23) into $S(X, Y)$, we have

$$\begin{aligned}S(X, Y)=e(Y)(H_1f-fH_1)X-e(X)(H_1f-fH_1)Y+u(Y)(H_2f-fH_2)X\\ -u(X)(H_2f-fH_2)Y+v(Y)(H_3f-fH_3)X-v(X)(H_3f-fH_3)Y\\ +(u(X)l_{12}(Y)+v(X)l_{13}(Y)-u(Y)l_{12}(X)-v(Y)l_{13}(X))E\\ +(e(Y)l_{12}(X)-v(Y)l_{23}(X)-e(X)l_{12}(Y)+v(X)l_{23}(Y))U\\ +(e(Y)l_{13}(X)+u(Y)l_{23}(X)-e(X)l_{13}(Y)-u(X)l_{23}(Y))V.\end{aligned} \tag{3.24}$$

If the connection of normal bundle of M is flat, then we can choose N_1, N_2 and N_3 in such a way that the 1-forms l_{ij} vanish identically. In this case, if f commute with H_i, that is,

$$H_if-fH_i=0, \quad (i=1, 2, 3) \tag{3.25}$$

then we get $S(X, Y)=0$. Thus we have

PROPOSITION 3.1. *Let M be a submaniold of codimension 3 of Kählerian manifold whose connection induced in the normal bundle is flat. If f ccmmute with H_i $(i=1, 2, 3)$,* *and* $\lambda_{12}=\lambda_{13}=0$, $\lambda_{23}=\lambda$ *in* (3.5)-(3.6), *then* M *admits a normal* (f, g, e, u, v, λ)-*structure.*

If M is a totally umbilical submanifold, then f commute with H_i. Thus we have

COROLLARY 3.2. *Let* M *be a totally umbilical submanifold of codimension* 3 *of a Kählerian manifold whose connection induced in the normal bundle is flat. If* $\lambda_{12}=\lambda_{13}=0$ *and* $\lambda_{23}=\lambda$ *in* (3.5)-(3.7), *then* M *admits a normal* (f, g, e, u, v, λ)-*structure.*

For a totally umbilical submanifold whose connection induced in the normal bundle is flat, we can choose unit normal vectors N_i $(i=1, 2, 3)$ in such a way that

$$h_i(X, Y)=h_i g(X, Y), \quad l_{ij}=0, \quad (i, j=1, 2, 3)$$

and consequently (3.21), (3.22) and (3.23) become

$$(\nabla_X e)Y=h_1 w(X, Y). \tag{3.26}$$

$$(\nabla_X u)Y=h_2 w(X, Y)-h_3 g(X, Y). \tag{3.27}$$

$$(\nabla_X v)Y=h_3 w(X, Y)+h_2 g(X, Y). \tag{3.28}$$

respectively. These equations give

$$(\nabla_X e)Y+(\nabla_Y e)X=0, \tag{3.29}$$

$$(\nabla_X u)Y+(\nabla_Y u)X=-2\lambda h_3 g(X, Y), \tag{3.30}$$

$$(\nabla_X v)Y+(\nabla_Y v)X=2\lambda h_2 g(X, Y), \tag{3.31}$$

which show that E defines an infinitesimal motion and U, V define infinitesimal conformal transformations respectirely.

§4. Manifold (I) with normal (f, g, e, u, v, λ)-structure.

In this section, we study a manifold with normal (f, g, e, u, v, λ)-structure satisfying certain condition.

Let M be a manifold with normal (f, g, e, u, v, λ)-structure. The structure being normal, we have

$$\begin{aligned} S(X, Y) = (\nabla_{fX} f)Y-(\nabla_{fY} f)X+f(\nabla_Y f)X-f(\nabla_X f)Y \\ +2de(X, Y)E+2du(X, Y)U+2dv(X, Y)V=0. \end{aligned} \tag{4.1}$$

We first prove

LEMMA 4.1. *In a manifold* M *with normal* (f, g, e, u, v, λ)-*structure, we have*

$$de(X, Y)=de(fX, fY), \tag{4.2}$$

$$\begin{aligned} 2du(X, Y)=2du(fX, fY)-\lambda(dv(fX, Y)+dv(X, fY))-((\nabla_{fX}\lambda)v(Y), \\ -(\nabla_{fY}\lambda)v(X)-\lambda((\nabla_X\lambda)u(Y)-(\nabla_Y\lambda)u(X)), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} 2dv(X, Y)=2dv(fX, fY)+\lambda(du(fX, Y)+du(X, fY)+((\nabla_{fX}\lambda)u(Y) \\ -(\nabla_{fY}\lambda)u(X)-\lambda((\nabla_X\lambda)v(Y)-(\nabla_Y\lambda)v(X)). \end{aligned} \tag{4.4}$$

Proof. Substituting (4.1) into $g(S(X, Y), E)=0$, we have

$$g((\nabla_{fX} f)Y, E)-g((\nabla_{fY} f)X, E)+2de(X, Y)=0,$$

or

$$-g(fY, \nabla_{fX}E) + g(fX, \nabla_{fY}E) + 2de(X, Y) = 0,$$

from which we have

$$-(\nabla_{fX}e)fY + (\nabla_{fY}e)fX + 2de(X, Y) = 0,$$

from which we have (4.2).

Next, substituting (4.1) into $g(S(X, Y), U) = 0$, we have

$$g((\nabla_{fX}f)Y, U) - g((\nabla_{fY}f)X, U) + \lambda(g((\nabla_Y f)X, V) - g((\nabla_X f)Y, V)) + 2(1-\lambda^2)du(X, Y) = 0,$$

or

$$(\nabla_{fX}\lambda)g(Y, V) + \lambda g(Y, \nabla_{fX}V) - g(fY, \nabla_{fX}U) - (\nabla_{fY}\lambda)g(X, V) - g(X, \nabla_{fY}V) + g(fX, \nabla_{fY}U) - \lambda((\nabla_Y\lambda)g(X, U) + \lambda g(X, \nabla_Y U) + g(fX, \nabla_Y V)) + \lambda((\nabla_X\lambda)g(Y, U) + g(Y, \nabla_X U) + g(fY, \nabla_X V)) + 2(1-\lambda^2)du(X, Y) = 0,$$

from which

$$\lambda(g(Y, \nabla_{fX}V) - g(fX, \nabla_Y V) - g(X, \nabla_{fY}V) + g(fY, \nabla_X V)) - 2du(fX, fY) + 2du(X, Y) + (\nabla_{fX}\lambda)g(Y, V) - (\nabla_{fY}\lambda)g(X, V) + \lambda((\nabla_X\lambda)g(Y, U) - (\nabla_Y\lambda)g(X, U)) = 0,$$

from which we have (4.3).

Similarly, computing $g(S(X, Y), V) = 0$, from (4.1) we have (4.4).

In a manifold with normal (f, g, e, u, v, λ)-structure, we put the condition

(4.5) $$de(X, Y) = w(X, Y).$$

As we have seen in the preceding section, for a submanifold of codimension 2 of a normal contact Riemannian manifold, we have

$$(\nabla_X e)Y = w(X, Y),$$

and consequently the condition (4.5) is always satisfied.

LEMMA 4.2. *Let M be a manifold with normal (f, g, e, u, v, λ)-structure satisfying (4.5). Then we have $\lambda = 0$ or $\lambda^2 = 1$.*

Proof. Substituting (4.5) into (4.2) and taking account of (1.1), we get

$$w(X, Y) = w(fX, fY) = -g(X, fY) + u(X)u(fY) + v(X)v(fY) = w(X, Y) + \lambda[u(X)v(Y) - v(X)u(Y)].$$

from which

$$\lambda[u(X)v(Y) - v(X)u(Y)] = 0.$$

Now putting $X = U$ and $Y = V$, we have

$$\lambda(1-\lambda^2)^2 = 0,$$

from which we have $\lambda = 0$ or $\lambda^2 = 1$.

PROPOSITION 4.3. *Let M be a manifold with normal (f, g, e, u, v, λ)-structure satisfying*

(4.5). *Then the manifold is either an almost normal contact Riemannian manifold or a manifold with f-structure of rank* $(2n-2)$.

Proof. M being a manifold with normal (f, g, e, u, v, λ)-structure satisfying (4.5). From Lemma 4.1, we have $\lambda^2=1$ or $\lambda=0$.

If $\lambda^2=1$, then $u(U)=v(V)=0$, and $U=V=0$. Thus we have

$$f^2X=-X+e(X)E.$$

Hence the manifold M is a normal contact Riemannian manifold.

If $\lambda=0$, then the vector fields U and V are unit vectors. From (1.3) and (1.4) we have

$$fU=0,\quad fV=0. \tag{4.6}$$

Operating f to (1.1) and using (4.6), we have

$$f^3X=-fX. \tag{4.7}$$

from which we have $f^3+f=0$ and rank of f is $2n$-2. Thus normal (f, g, e, u, v, λ)-structure is an f-structure of rank $2n-2$.

§ 5. Manifold (II) with normal (f, g, e, u, v, λ)-structure

In this section, we assume that

$$du(X, Y)=w(X, Y). \tag{5.1}$$

and

$$dv(X, Y)=\phi w(X, Y). \tag{5.2}$$

where ϕ is a differentiable function.

LEMMA 5.1. *Let M be a manifold with normal* (f, g, e, u, v, λ)*-structure satisfying* (5.1) *and* (5.2). *If the function* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero, then we have*

$$\nabla_V\lambda=-(1-\lambda^2), \tag{5.3}$$

$$\nabla_U\lambda=\phi(1-\lambda^2). \tag{5.4}$$

Proof. Substituting $X=U$ and $Y=V$ into (4.3), we get

$$2du(U, V)=2du(fU, fV)-\lambda(dv(fU, V)+dv(U, fV))-(1-\lambda^2)(\nabla_{fU}\lambda)$$
$$+\lambda(1-\lambda^2)(\nabla_V\lambda),$$

from which, using (1.3)

$$2(1-\lambda^2)du(U, V)-2\lambda(1-\lambda^2)(\nabla_V\lambda)=0. \tag{5.5}$$

On the other hand, from (5.1) we get

$$du(U, V)=w(U, V)=g(fU, V)=-\lambda(1-\lambda^2). \tag{5.6}$$

Substituting (5.6) into (5.5), we have

$$-2\lambda(1-\lambda^2)^2-2\lambda(1-\lambda^2)(\nabla_V\lambda)=0,$$

from which we have (5.3).

Similarly, Substituting $X=U$ and $Y=V$ into (4.4), we have (5.4).

LEMMA 5.2. *Under the same assumptions as those in Lemma* 5.1, *we have*

(5.7) $$\nabla_X\lambda=\phi u(X)-v(X).$$

Proof. *From* (4.3), (5.1) and (5.2), we get

$$2w(X, Y)=2w(fX, fY)-(\nabla_{fX}\lambda)v(Y)+(\nabla_{fY}\lambda)v(X)-\lambda(\nabla_X\lambda)u(Y)+\lambda(\nabla_Y\lambda)u(X),$$

or

$$2\lambda(u(X)v(Y)-v(X)u(Y))-(\nabla_{fX}\lambda)v(Y)+(\nabla_{fY}\lambda)v(X)-\lambda(\nabla_X\lambda)u(Y)+\lambda(\nabla_Y\lambda)u(X).$$

Replacing X by U in the last equation, we find

$$2\lambda(1-\lambda^2)v(Y)+\lambda(\nabla_V\lambda)v(Y)-\lambda(\nabla_U\lambda)u(Y)+\lambda(1-\lambda^2)(\nabla_Y\lambda)=0,$$

from which, substituting (5.3) and (5.4),

$$2\lambda(1-\lambda^2)\ v(Y)-\lambda(1-\lambda^2)v(Y)-\phi\lambda(1-\lambda^2)u(Y)+\lambda(1-\lambda^2)(\nabla_Y\lambda)=0,$$

which proves (5.7).

LEMMA 5.3. *Under the same assumptions as those in Lemma* 5.1, ϕ *is constant.*

Proof. Differentiating (5.7) covariantly, we have

$$g(\nabla_Y\nabla\lambda, X)=(\nabla_Y\phi)u(X)+\phi(\nabla_Yu)X-(\nabla_Yv)X.$$

Replacing X by Y in the last equation, and subtracting the original one, we have

$$(\nabla_Y\phi)u(X)-(\nabla_X\phi)u)Y+\phi((\nabla_Yu)X-(\nabla_Xu)Y)-((\nabla_Yv)X-(\nabla_Xv)Y)=0,$$

from which, using (5.1) and (5.2)

$$(\nabla_Y\phi)u(X)=(\nabla_X\phi)u(Y).$$

which implies that

(5.8) $$\nabla_X\phi=\alpha u(X),$$

for some scalar function α.

Differentiating (5.8) covariantly, we get

$$g(\nabla_Y\nabla\phi, X)=(\nabla_Y\alpha)u(X)+\alpha(\nabla_Yu)X.$$

Replacing X by Y in the last equation, we get

$$(\nabla_Y\alpha)u(X)-(\nabla_X\alpha)u(Y)+\alpha((\nabla_Yu)X-(\nabla_Xu)Y)=0,$$

from which, using (5.1),

(5.9) $$2\alpha w(X, Y)=(\nabla_Y\alpha)u(X)-(\nabla_X\alpha)u(Y).$$

Thus we have $\alpha=0$ because the rank of w is almost everywhere maximum. This shows that ϕ is constant.

LEMMA 5.4. *Under the same assumptions as those in Lemma* 5.1, *we have*

(5.10) $$(\nabla_Xu)U+(\nabla_Uu)X=-2\lambda\phi u(X),$$

and

(5.11) $$(\nabla_Xv)V+(\nabla_Vv)X=2\lambda v(X).$$

Proof. Differentiating $u(U)=1-\lambda^2$ covariantly and using (5.7), we find

$$2(\nabla_Xu)U=-2\lambda(\phi u(X)-v(X)).$$

Substituting this into

$$2(\nabla_X u)U = ((\nabla_X u)U + (\nabla_U u)X) + ((\nabla_X u)U - (\nabla_U u)X)$$
$$= ((\nabla_X u)U + (\nabla_U u)X) + 2\lambda v(X),$$

we find

$$-2\lambda(\phi u(X) - v(X)) = (\nabla_X u)U + (\nabla_U u)X + 2\lambda v(X),$$

from which we have (5.10).

Similarly, we can prove (5.11).

LEMMA 5.5. *Under the same assumptions as those in Lemma* 5.1, *we have*

(5.12) $$(\nabla_Z w)(fY, X) - \nabla_Z w)(fX, Y) = -e(X)(\nabla_Y e)Z + e(Y)(\nabla_X e)Z$$
$$-u(X)(\nabla_Y u)Z + u(Y)(\nabla_X u)Z - v(X)(\nabla_Y v)Z + v(Y)(\nabla_X v)Z.$$

Proof. Since $w(X, Y)$ is given by

$$2w(X, Y) = (\nabla_X u)Y - (\nabla_Y u)X,$$

we have $dw=0$, that is,

(5.13) $$(\nabla_Z w)(X, Y) + (\nabla_X w)(Y, Z) + (\nabla_Y w)(Z, X) = 0.$$

On the other hand, computing $g(S(X, Y), Z) = 0$, we obtain

$$(\nabla_{fX} w)(Y, Z) - (\nabla_{fY} w)(X, Z) + w((\nabla_Y f)X, Z) - w((\nabla_X f)Y, Z)$$
$$+2de(X, Y)e(Z) + 2du(X, Y)u(Z) + 2dv(X, Y)v(Z) = 0,$$

from which, using (5.13),

$$-(\nabla_Y w)(Z, fX) - (\nabla_Z w)(fX, Y) + (\nabla_X w)(Z, fY) + (\nabla_Z w)(fY, X)$$
$$+w((\nabla_Y f)X, Z) - w((\nabla_X f)Y, Z) + 2de(X, Y)e(Z) + 2du(X, Y)u(Z)$$
$$+2dv(X, Y)v(Z) = 0,$$

that is,

$$(\nabla_Z w(fY, X) - (\nabla_Z w)(fX, Y) - \nabla_Y w(Z, fX) + \nabla_X w(Z, fY) + w(\nabla_Y Z, fX)$$
$$+w(Z, f(\nabla_Y X)) - w(\nabla_X Z, fY) - w(Z, f(\nabla_X Y)) + 2de(X, Y)e(Z)$$
$$+2du(X, Y)u(Z) + 2dv(X, Y)v(Z) = 0.$$

Substituting

$$w(Z, fX) = g(X, Z) - e(X)e(Z) - u(X)u(Z) - v(X)v(Z),$$

we find

$$(\nabla_Z w)(fY, X) - (\nabla_Z w)(fX, Y) - e(Y)(\nabla_X e)Z + e(X)(\nabla_Y e)Z - u(Y)(\nabla_X u)Z$$
$$+u(X)(\nabla_Y u)Z - v(Y)(\nabla_X v)Z = 0,$$

which proves (5.12).

LEMMA 5.6. *Under the same assumptions as those in Lemma* 5.1, *we have*

(5.14) $$(\nabla_Z u)Y + (\nabla_Y u)Z = -2\lambda\phi g(Z, Y) + 2((\nabla_Z u)E + \lambda\phi e(Z))e(Y),$$

(5.15) $$(\nabla_Z v)Y + (\nabla_Y v)Z = 2\lambda g(Z, Y) + 2((\nabla_Z v)E - \lambda e(Z))e(Y).$$

Proof. Substituting U for X and using (1.3), we get

$$(\nabla_Z w)fY, U) - (\nabla_Z w)(fU, Y) = e(Y)(\nabla_U e)Z - (1-\lambda^2)(\nabla_Y u)Z$$

$$+u(Y)(\nabla_U u)Z+v(Y)(\nabla_U v)Z,$$

from which

$$\nabla_Z w(fY, U)-\nabla_Z w(fU, Y)+w((\nabla_Z f)U, Y)-w((\nabla_Z f)Y, U)$$
$$+e(Y)(\nabla_U e)Z-(1-\lambda^2)(\nabla_Y u)Z+u(Y)(\nabla_U u)Z+v(Y)(\nabla_U v)Z. \tag{5.14}$$

On the other hand, from (1.12) we find

$$\nabla_Z w(fY, U)=\nabla_Z w(fU, Y), \tag{5.15}$$

and using (1.4)

$$w((\nabla_Z f)U, Y)=-g(fY, (\nabla_Z f)U)$$
$$=g(fY, (\nabla_Z \lambda)V)+\lambda g(fY, \nabla_Z V)+g(fY, f(\nabla_Z U)). \tag{5.16}$$

Differentiating

$$g(fU, fY)=-\lambda v(fY)=\lambda^2 u(Y)$$

covariantly, we have

$$w((\nabla_Z f)Y, U)=-g(fY, (\nabla_Z \lambda)V)-\lambda g(fY, \nabla_Z V)-\lambda^2(\nabla_Z u)Y. \tag{5.17}$$

Substituting (5.15), (5.16) and (5.17) into (5.14), we have

$$2\lambda(\nabla_Z v)fY+(\nabla_Z u)Y+e(Y)(\nabla_Z e)U-u(Y)(\nabla_Z u)U+v(Y)(\nabla_Z v)U+\lambda^2(\nabla_Z u)Y$$
$$=e(Y)(\nabla_U e)Z-(1-\lambda^2)(\nabla_Y u)Z+u(Y)(\nabla_U u)Z+v(Y)(\nabla_U v)Z,$$

from which

$$2\lambda(\nabla_Z v)fY+(\nabla_Z u)Y+(\nabla_Y u)Z+\lambda^2((\nabla_Z u)Y-(\nabla_Y u)Z$$
$$=e(Y)((\nabla_U e)Z-(\nabla_Z e)U)+u(Y)((\nabla_U u)Z+(\nabla_Z u)U)+v(Y)((\nabla_U v)Z-(\nabla_Z v)U),$$

or

$$2\lambda(\nabla_Z v)fY+((\nabla_Z u)Y+(\nabla_Y u)Z)+2\lambda^2 w(Z, Y)$$
$$=e(Y)((\nabla_U e)Z-(\nabla_Z e)U)+u(Y)((\nabla_U u)Z+(\nabla_Z u)U)+2\phi v(Y)w(U, Z).$$

Substituting

$$2(\nabla_Z v)fY=((\nabla_Z v)fY+(\nabla_{fY} v)Z)+((\nabla_Z v)fY-(\nabla_{fY} v)Z)$$
$$=(\nabla_Z v)fY+(\nabla_{fY} v)Z+2\phi w(Z, fY),$$

and (5.10) into the equation above, we get

$$\lambda((\nabla_Z v)fY+(\nabla_{fY} v)Z)+2\lambda\phi(g(Y, Z)-e(Y)e(Z)-u(Y)u(Z)-v(Y)v(Z))$$
$$+((\nabla_Z u)Y+(\nabla_Y u)Z)+2\lambda^2 w(Z, Y)$$
$$=e(Y)((\nabla_U e)Z-(\nabla_Z e)U)-2\lambda\phi u(Y)u(Z)+2\lambda\phi v(Y)v(Z),$$

from which

$$(\nabla_Z u)Y+(\nabla_Y u)Z=\lambda((\nabla_Z v)fY+(\nabla_{fY} v)Z)-2\lambda\phi g(Z, Y)-2\lambda^2 w(Z, Y)$$
$$+2\lambda\phi e(Y)e(Z)+e(Y)((\nabla_U e)Z-(\nabla_Z e)U). \tag{5.18}$$

Similarly, we have

$$(\nabla_Z v)Y+(\nabla_Y u)Z=\lambda((\nabla_Z u)fY+(\nabla_{fY} u)Z)+2\lambda g(Z, Y)-2\lambda^2\phi w(Z, Y)$$
$$-2\lambda e(Y)e(Z)+e(Y)((\nabla_V e)Z-(\nabla_Z e)V). \tag{5.19}$$

Substituting (5.19) into (5.18) and using (5.10), we obtain

$$(5.20)\quad (1-\lambda^2)\left((\nabla_Z u)Y+(\nabla_Y u)Z\right)=-2\lambda(1-\lambda^2)\phi g(Z,Y)-2\lambda^3\phi v(Y)v(Z)$$
$$-\lambda^2 v(Y)\left((\nabla_Z u)V+(\nabla_V u)Z\right)$$
$$+(1-\lambda^2)\left((\nabla_U e)Z-(\nabla_Z e)U+2\lambda\phi e(Z)\right)e(Y).$$

On the other hand, substituting E for Y in (5.18), we get

$$(5.21)\quad (\nabla_Z u)E+(\nabla_E u)Z=(\nabla_U e)Z-(\nabla_Z e)U,$$

and taking account of (5.1),

$$2du(Z,E)=(\nabla_Z u)E-(\nabla_E u)Z=0,$$

from which, (5.21) is written by

$$(5.22)\quad 2(\nabla_Z u)E=(\nabla_U e)Z-(\nabla_Z e)U.$$

Substituting V for Y in (5.20), we obtain

$$(5.23)\quad (\nabla_Z u)V+(V_V u)Z=-2\nabla\phi v(Z).$$

Consequently, substituting (5.22) and (5.23) into (5.20), we have (5.14). And substituting (5.14) into (5.19), we have (5.15).

If we put

$$(5.24)\quad e(\nabla_Z U)=-\lambda\phi e(Z),\quad e(\nabla_Z V)=\lambda e(Z),$$

in (5.14) and (5.15), we have respectively

$$(5.25)\quad (\nabla_Z u)Y+(\nabla_Y u)Z=-2\lambda\phi g(Z,Y),$$

$$(5.26)\quad (\nabla_Z v)Y+(\nabla_Y u)Z=2\lambda g(Z,Y).$$

Equations (5.25) and (5.26) show that the vector fields U and V define infinitesimal conformal transformations respectively.

In this case, using (5.1) and (5.2), we have

$$(5.27)\quad (\nabla_Z u)Y=-\lambda\phi g(Z,Y)+w(Z,Y),$$

$$(5.28)\quad (\nabla_Z v)Y=\lambda g(Z,Y)+\phi w(Z,Y).$$

Thus we have

PROPOSITION 5.7. *Let M be a manifold with normal (f, g, e, u, v, λ) -structure satisfying* (5.1), (5.2) *and* (5.24). *If the function* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero, then the vector fields U and V define infintesimal conformal tsansformations respectively.*

§6. Odd dimensional spheres

We prove

THEOREM 6.1. *Let M be a complete manifold with normal (f, g, e, u, v, λ)-structure satisfying* (5.1), (5.2) *and* (5.24). *If the function* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero, then M is isometric with an odd dimensional sphere.*

Proof. Differentiating (5.7) covariantly, we have

$$g(\nabla_Y(\nabla\lambda),X)=\phi(\nabla_Y u)X-(\nabla_Y v)X$$

ϕ being a constant, from which, using (5.14) and (5.15),

$$(6.1)\quad g(\nabla_Y(\nabla\lambda),X)=-(1+\phi^2)\lambda g(X,Y).$$

Thus, by means of Obata's theorem, M is isometric with a sphere.

Next, we prove

THEOREM 6.2. *Let M be a complete manifold with normal (f, g, e, u, v, λ)-structure satisfying* (5.24) *and*

$$(\nabla_X u) Y = w(X, Y). \tag{6.2}$$

If $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then M is an odd dimensional sphere.

Proof. Diffrentiating $u(U)=1-\lambda^2$ covariantly, we have

$$(\nabla_X u) U = -\lambda \nabla_X \lambda,$$

from which, substituting Y for U in (6.2), we get

$$\nabla_X \lambda = -v(X). \tag{6.3}$$

This shows that

$$(\nabla_X v) Y - (\nabla_Y v) X = 0. \tag{6.4}$$

Equation (6.2) shows that

$$(\nabla_X u) Y - (\nabla_Y u) X = 2w(X, Y). \tag{6.5}$$

Equations (6.4) and (6.5) satisfy the equations (5.1) and (5.2) respectively, and consequently M is isometric with an odd dimensional sphere.

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