A NATURAL APPROACH TO FREDHOLM STRUCTURES ON BANACH MANIFOLDS

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Introduction. The author studies in this paper to link the Fredholm structures of Banach spaces and differential structures on Banach manifolds more inclusively. The approaching should be a natural way and abstracted in the following three parts.

Although some elementary analytical approach to the Fredholm structures of Banach spaces is discussed in [1] with detail self-contained direct method, and some general discussion of it is found in [6], in Part A, however, is used the notion of the principal fibre bundle [3] to produce the same Fredholm structures, and the author thinks that it is more natural.

In Part B the author defines the structures that cover all smooth Banach manifolds, and reduces them to $\Theta$-structures as a particular case.

The theorem in Part C gives a characterization of $\Phi(M, E)$, the set of all $\Theta$-admissible maps of $\Theta$-manifold $M$ into $E$, the Banach space which models $M$. The argument of this part should refer to [5] heavily.

A. Let $X$ be a topological space and $G$ a topological group acting continuously on the right of $X$, i.e., the right action $X \times G \to X$, defined by $(x, g) \mapsto x \cdot g$ such that $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$ and $x \cdot 1 = x$ for all $x$ in $X$ and $g_1, g_2$ in $G$, where $1$ is the identity in $G$, is continuous.

A $G$-bundle or a principal fibre bundle with the structural group $G$ is the surjective map $p: X \to B$ of $X$ onto the orbit space $B = X / G$ with the quotient topology which assigns to each $x$ in $X$ its orbit $p(x) \in B$ under the $G$-action such that the $G$-action is principal which means that: (1) The $G$-action is free, i.e., $x \cdot g = x$ only when $g = 1$, (2) it is proper, i.e., the map $\theta: A \to G$ defined by $\theta(x, x \cdot g) = g$ is continuous, where $A = \{(x, x') \in X \times X | x' = x \cdot g$ for some (unique) $g \in G\}$ and (3) it is locally trivial, i.e., each orbit $b$ in $B$ has a neighborhood on which there exists a continuous section. We call the acting group $G$ the structural group of the $G$-bundle.

We denote by $E$ an infinite dimensional Banach space, $L(E)$ the Banach space of all continuous linear maps of $E$ into itself with the Sup. norm, and $C(E)$ the closed subspace of $L(E)$ consisting of compact operators [1]. If we think of $L(E)$ as a Banach Lie algebra with unit $I$, the identity operator, and bracket $[T, S] = T \cdot S - S \cdot T$ for $T, S \in L(E)$, then $C(E)$ is a bilateral ideal without unit of $L(E)$. Let the canonical homomorphism $\phi: L(E) \to L(E) / C(E)$ be the topological identification. We shall find the structural group $G$ with which $\phi$ becomes a principal fibre bundle.

Theorem. The structural group $G$ that makes $\phi: L(E) \to L(E) / C(E)$ into a $G$-bundle is contained in the subset $\{I + k | k \in C(E)\}$ of $L(E)$.

Proof. Since $G$ must act on the right of $L(E)$ continuously, the map $L(E) \times G \to L(E)$
defined by \((T, g) \rightarrow T \cdot g\) is continuous, and the orbit space \(L(E)/G\) of the group \(G\) must be equal to \(L(E)/C(E)\).

For any \(T\) in \(L(E)\) the orbit \(\rho(T)\) of \(T\) under the \(G\)-action must be expressed by \(\rho(T) = \{T \cdot g | g \in G\}\). On the other hand, by the canonical homomorphism \(\rho, \rho(T) = T + C(E)\). Thus \(T + C(E) = \{T \cdot g | g \in G\}\). Hence for any \(g \in G\) there exists a \(k\) in \(C(E)\) such that \(T + k = T \cdot g\) and this equation holds if the binary operation is considered as the composition and \(g\) as \(1 + k\) for some \(k'\) in \(C(E)\). Therefore \(g\) belongs to the set \(\{I + k | k \in C(E)\}\).

We can easily see that with this \(C(E)\) the \(\pi: GL(E) \rightarrow GL(E)/GL_c(E)\) is a \(GL_c(E)\)-bundle as the canonical homomorphism.

The structural group \(G\) found in the above theorem within \(L_c(E) = \{I + k | k \in C(E)\}\) is called the Eichholz group of \(E\) and denoted by \(GL_c(E)\). The author contends that the above theorem has produced the group \(GL_c(E)\) by a natural way.

Denote \(GL(E) = \{T \in L(E) | T\) has its inverse\}. Then \(GL(E)\) is an open group (under composition) in \(L(E)\) and \(GL_c(E)\) is an invariant subgroup of \(GL(E)\), and hence \(\pi: GL(E) \rightarrow GL(E)/GL_c(E)\) is a \(GL_c(E)\)-bundle as the canonical homomorphism.

On the other hand denote \(\Phi(E) = \{T \in L(E) | \dim \ker T, \dim \coker T < \infty\}\) and an element of \(\Phi(E)\) is called a Fredholm operator of \(E\). Defining index of \(T\) by \(\text{ind } \Phi(E) \rightarrow \mathbb{Z}\) of \(\Phi(E)\) into integers set \(\mathbb{Z}\) and denote \(\Phi_0(E) = \text{ind}^{-1}(0)\). If we denote \(G_0 = \rho(\Phi_0(E))\), then \(\rho: \Phi_0(E) \rightarrow G_0\) is a \(GL_c(E)\)-bundle as the subbundle of \(\rho: L(E) \rightarrow L(E)/C(E)\) induced by the restriction. \(G_0\) is a group isomorphic with \(GL(E)/GL_c(E)\) and the elementary detail proof of it is given in [1]. With this isomorphism \(\gamma\) we have the commutative diagram

\[
\begin{array}{ccc}
GL(E) & \xrightarrow{i} & \Phi(E) \\
\pi \downarrow & & \gamma \downarrow \\
GL(E)/GL_c(E) & \rightarrow & G_0
\end{array}
\]

where \(i\) is the inclusion. We call \(G_0 = GL(E)/GL_c(E)\) the Fredholm structure of \(E\). We notice that \(\rho: \Phi_0(E) \rightarrow G_0\) is a homotopy equivalence, since its fibres are contractible.

**B.** We will denote by \(M\) a smooth (infinite differentiable) manifold modelled on \(E\), denoted by \(E\)-manifold. We define that \(M\) admits a \(\Phi\)-structure if there is a collection \(\Phi_M\) of charts \((U_i, Q_i)\) covering \(M\) and satisfying that for all \(i, j\) and \(x \in U_i \cap U_j\),

\[
D(\theta_j \theta_i^{-1})(\theta_i(x)) \in \Phi(E),
\]

where \(D\) denotes the differentiation operator. A member of \(\Phi_M\) is often called \(\Phi\)-chart.

If \((V, \varphi)\) is a differentiable chart of \(M\) such that for each \(x \in V\), there exists a \(\Phi\)-chart \((U_i, \theta_i)\) so that \(D(\varphi \theta_i^{-1})(\theta_i(x)) \in \Phi(E)\), then \(D(\varphi \theta_j^{-1})(\theta_j(x)) = D(\varphi \theta_i^{-1}) \circ D(\theta_i \theta_j^{-1})(\theta_j(x)) \in \Phi(E)\) for all \(j\) with \(U_i \cap U_j \neq \emptyset\) and \(x \in U_i \cap U_j\). Therefore we can always construct a maximal collection \(\Phi_M\) which is meant by \(\Phi\)-structure admitted on \(M\), and \(M\) with \(\Phi_M\) is called a Fredholm manifold, denoted by \(\Phi\)-manifold.

We will apply the similar notations from \(\Phi(E)\) to \(GL_c(E)\) by using \(\Theta_M\) and \(E\)-manifold, etc.

**THEOREM.** Every smooth manifold modelled on any Banach space is a \(\Phi\)-manifold.
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Proof. If we recall the definition of the tangent space $TM_x$ of $M$ at $x$, a tangent vector at $x \in M$ is an equivalence class of all triples $(U_i, \theta_i, v)$, where the equivalence relation $(U_i, \theta_i, v) \sim (U_j, \theta_j, w)$ is given by the condition $D(\theta_j \circ \theta_i^{-1})(\theta_i(x))v = w$ for $v \in E_i$, $w \in E_j$ (with $E_i = E_j = E$). We can easily see that $D(\theta_j \circ \theta_i^{-1})(\theta_i(x)) = D(\theta_j \circ \theta_i^{-1})(\theta_j(x))$, and so in fact $D(\theta_j \circ \theta_i^{-1})(\theta_i(x)) \in GL(E) \subset \Phi(E) \subset \Phi(E)$ for all $i, j$ and $x \in U_i \cap U_j$.

Corollary. Any coordinate transformation $\theta_j \circ \theta_i^{-1}$ on $U_i \cap U_j \subset M$ has its derivative at $x \in U_i \cap U_j$ expressible as $D(\theta_j \circ \theta_i^{-1})(\theta_i(x)) = T + k$, where $T \in GL(E)$ and $k \in \Phi(E)$.

Proof. Since $D(\theta_j \circ \theta_i^{-1})(\theta_i(x)) \in GL(E)$, there exists some $T$ in $GL(E)$ such that $D(\theta_j \circ \theta_i^{-1})(\theta_i(x))$ is contained in the orbit of $T$ under the Fredholm group $GLC(E)$, i.e., contained in $T \cdot GLC(E) \subset T + C(E)$.

Due to the last corollary, $\Theta$-manifold is the manifold $M$ that admits Fredholm structure $\Theta_M$ such that all coordinate transformations have their derivatives contained in the orbit of $I \subset GL(E)$ under the action group $GLC(E)$, We can therefore give the following remark as a consequence.

Remark. The tangent bundle $\pi: TM \to M$ is a $GL(E)$-bundle for any smooth $E$-manifold, and in particular it is a $GLC(E)$-bundle for any $\Theta$-manifold modelled on $E$.\]

C. Let $M$ and $N$ be smooth manifolds modelled on Banach spaces $E$ and $F$ respectively, and $f: M \to N$ be a smooth map. $f$ is called a Fredholm map if its differential $f_\#: TM \to TN$ at each $x$ in $M$ is a Fredholm operator of $TM_x$ into $TN_{f(x)}$, and denoted by $\Phi$-map. We define the index of $f$ by that of $f_\#(x)$ when $M$ is connected. We therefore assume that $M$ is connected whenever this notion is used.

One of nice visible non-trivial examples of $\Phi$-map the index of which is purposely pursued is found in [7] with an infinite dimensional Hilbert manifold.

Now let $M$ and $N$ be $\Theta$-manifolds modelled on the same Banach space $E$. A map $f: M \to N$ is said to be $\Theta$-admissible if the vector bundle map $Df: TM \to TN$ has the form $D(f_\#(x), v) = (f(x), v + k(x)v)$ with $k(x) \in \Phi(E)$ for each $x \in M$ and $v$ in $TM_x$, where $TM$ and $TN$ are given the $\Theta$-structure induced from $M$ and $N$ respectively.

If we refer to [6], the index function $\text{ind}: \Phi(E) \to \mathbb{Z}$ has the property that $\text{ind}(T+k) = \text{ind} T$ for any $T$ in $\Phi(E)$ and $k$ in $\Phi(E)$. Hence $Df_\# = I + k(x): TM_x \to TN_{f(x)}$ has index zero at each $x \in M$. Therefore we have the following:

Remark. Any $\Theta$-admissible map is a $\Phi$-map, a Fredholm map of index zero.

The following lemma is a particular case of so-called "the pull-back $\Theta$-structure $\{M, N, f\}_\Theta$ on $M"$ of a $\Phi$-map $f: M \to N$, where $M$ and $N$ are $\Theta$-manifolds modelled on $F$ and $E$ respectively (Theorem 2.2 in [5]). In the sequel $E$ is understood to have the $\Theta$-canonical $\Theta$-structure.

Lemma. If $M$ is a $\Theta$-manifold with $\Theta_M$ modelled on $E$ and admits a smooth partitions
of unity, then there is a $\Phi^0$-map $f: M \rightarrow E$ such that $f$ is $\Theta$-admissible.

Proof. Let $\{U_i, \theta_i\}$ be the coordinate covering of $M$ by admissible charts and $\{\gamma_i\}$ be the partitions of unity subordinate to $\{U_i\}$. Define $f: M \rightarrow E$ by $f(x) = \sum \theta_i(x) \gamma_i(x) \cdot v$.

Then for any chart $(U, \theta)$ in $\Theta_M$ with $\theta(x) = 0$,

$$D(f\theta^{-1}) \omega = \sum \gamma_i(x) D(\theta(\theta^{-1}) \omega) + \sum_i (D(\gamma_i \theta^{-1}) \gamma_i(x) \cdot v)$$

and the maps of the first and second terms belong to $L_C(E)$ and $C(E)$ respectively. Therefore we can see that $f$ is $\Theta$-admissible if we change the right side into the form $I + k(x)$ as a consequence.

Denote by $\Theta(M, E)$ the set of all $\Theta$-admissible maps of $M$ into $E$. Then by this lemma it is non-empty, and by that remark it is a subset of $\Phi_0(M, E)$, the set of all $\Phi_0$-maps of $M$ into $E$.

For any pair $f, g$ in $\Phi_0(M, E)$, $f$ and $g$ are said to be $\Phi_0$-homotopic if there exists a homotopy $h: M \times [0,1] \rightarrow E$ such that for each $t$ in $[0,1]$, $h_t: M \rightarrow E$ are $\Phi_0$-maps with $h_0 = f$ and $h_1 = g$, called $\Phi_0$-homotopy. We denote by $\Phi_0[M, E]$ the $\Phi_0$-homotopic equivalence classes in $\Phi_0(M, E)$.

Denoting by $[M, \Phi_0(E)]$ the homotopy classes of all continuous mappings of $M$ into $\Phi_0(E)$, we have the bijection $\rho: \Phi_0[M, E] \rightarrow [M, \Phi_0(E)]$ defined by $\rho[f] = [Df\circ \tau]$, where $(Df\circ \tau) (x) = Df \circ \tau_x$, $[\cdot]$ denotes the equivalence class and $\tau$ the trivialization $: M \times E \rightarrow TM$ after a group structure is induced in $[M, \Phi_0(E)]$ by the $GL_c(E)$-bundle $\rho: \Phi_0(E) \rightarrow G_0$ (due to the Proposition 2.4 in [5]).

THEOREM. $\Theta(M, E)$ is exactly the same class as $[f]$, where $f$ is the $\Phi_0$-map of $M$ into $E$ constructed in the proof of the preceding lemma.

Proof. Let $f \in \Theta(M, E)$ be the particular map constructed in the proof of the preceding lemma. It is clear that $[f]$ is mapped onto the identity of $[M, \Phi_0(E)]$ by $\rho$ and also it is obvious that for any $g \in \Theta(M, E)$, $Df \circ \tau_x = Dg \circ \tau_x \in C(E)$. Hence $\rho Df \circ \tau_x = \rho Dg \circ \tau_x \in G_0$. Since $\rho: \Phi_0(E) \rightarrow G_0$ is the homotopy equivalence, $[Df \circ \tau] = [Dg \circ \tau]$. Now by the bijection $\rho$, $[g] = [f]$, i.e., $g \in [f]$. which implies that $\Theta(M, E) \subset [f]$.

Conversely if $g \in [f]$, $[Dg \circ \tau]$ is the identity of $[M, \Phi_0(E)]$ and so $(\rho Dg \circ \tau)(x) = I$. $GL_c(E) \subseteq GL(E) / GL_c(E)$ by the commutative diagram in Part A, i.e., $Dg \circ \tau_x = I + k(x)$, where $k: M \rightarrow C(E)$, which implies that $[f] \subseteq \Theta(M, E)$. This completes the proof.

References.

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