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ON CONVERGENCE OF SEMIGROUPS OF OPERATORS IN BANACH SPACES

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Let X be a real Banach space. A family $\{T(t)\,;\ t\geq 0\}$ of operators from a subset X_0 of X into itself is called a semigroup of type ω on X_0 if the following conditions are satisfied:

$$T(0)=I,\quad T(t+s)=T(t)\,T(s)\ \text{ for } t,s\geq 0,$$
$$\lim_{t\to 0^+} T(t)\,x=x \text{ for } x\in X_0$$

and there exists a real number ω such that

(1) $$\|T(t)\,x-T(t)\,y\|\leqslant e^{\omega t}\,\|x-y\|$$

for $t\geq 0$ and $x,y\in X_0$. We define the infinitesimal generator A_0 of a semigroup $\{T(t)\,;\ t\geq 0\}$ of type ω on X_0 by

$$A_0x=\lim_{t\to 0^+} t^{-1}\{T(t)\,x-x\}$$

for $x\in X_0$ whenever the right side exists. A subset A of $X\times X$ is said to be in the class $\mathcal{A}(\omega)$, $\omega\geq 0$, if for $0<\lambda\omega<1$ and for $[x_k,\ y_k]\in A$, $k=1,2$, we have

(2) $$\|(x_1+\lambda y_1)-(x_2+\lambda y_2)\|\geq(1-\lambda\omega)\,\|x_1-x_2\|.$$

We say that A is *accretive* if $\omega=0$, and in addition, A is *m-accretive* if $R(I+\lambda A)=X$ for all $\lambda>0$.

Put $J_\lambda=(I+\lambda A)^{-1}$ and $A_\lambda=\lambda^{-1}(I-J_\lambda)$ for $\lambda>0$. The next lemma is well-known:

LEMMA A. *Let $A\in\mathcal{A}(\omega)$, $\omega\geq 0$ and let $0<\lambda\omega<1$. Then*

1) *J_λ is a function and*
$$\|J_\lambda x-J_\lambda y\|\leqslant(1-\lambda\omega)^{-1}\|x-y\|\quad \textit{for } x,y\in D(J_\lambda),$$

2) *A_λ is a function in the class $\mathcal{A}\{\omega(1-\lambda\omega)^{-1}\}$ and*
$$|AJ_\lambda x|\leqslant\|A_\lambda x\|\leqslant(1-\lambda\omega)^{-1}|Ax|\quad \textit{for } x\in D(J_\lambda)\cap D(A)$$

where $|Ax|=\inf\{\|y\|\,;\ y\in Ax\}$.

In the previous paper [5] the author proved the following Theorem C with making use of Lemma B.

LEMMA B. *Let $A\subset\mathcal{A}(\omega)$, $\omega\geq 0$ with* $R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ and *let* $0<\lambda\omega<1/2$. *Then $-A_\lambda$ is the infinitesimal generator of a semigroup $\{T_\lambda(t)\,;\ t\geq 0\}$ of type $\omega(1-\lambda\omega)^{-1}$ on $\overline{\mathrm{co}D(A)}$ which satisfies the following theree conditions:*

(i) $u_\lambda(t)=T_\lambda(t)\,x$ for $x\in\overline{\mathrm{co}D(A)}$ *is a unique solution of the Cauchy problem*

$$\begin{cases} du_\lambda(t)/dt+A_\lambda u_\lambda(t)=0\\ u_\lambda(0)=x,\end{cases}$$

(ii) *Furthermore*

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$$\| T_\lambda(n\lambda)x - J_\lambda^n x \|$$
(3) $$\leqslant (1-\lambda\omega)^{-n} e^{n\lambda\omega/(1-\lambda\omega)} \{n^2\lambda^2\omega^2(1-\lambda\omega)^{-2} + n\lambda\omega(1-\lambda\omega)^{-1} + n\}^{1/2} \| J_\lambda x - x \| .$$

(iii) *There exists $h_0>0$ depending on $\lambda>0$ and $x\in \overline{\mathrm{co}D(A)}$ such that*

(4) $$\| T_\lambda(t+h)x - T_\lambda(t)x \| \leqslant h e^{\omega t/(1-\lambda\omega)} (\| A_\lambda x \| + 1) \quad \text{for } 0<h<h_0.$$

Since $(1-s)^{-n} \leqslant e^{2ns}$ for $s\in[0, 1/2]$, (3) implies that

(5) $$\| T_\lambda(n\lambda)x - J_\lambda^n x \| \leqslant \sqrt{\lambda} K(t,\lambda,\omega) \| A_\lambda x \| \leqslant \sqrt{\lambda}(1-\lambda\omega)^{-1} K(t,\lambda,\omega) |Ax|$$

for n such that $t=n\lambda+\delta$, $0\leqslant\delta<\lambda$, and for $x\in D(A)$, where

$$K(t,\lambda,\omega) = e^{(2\omega+1/(1-\lambda\omega))t} \{\lambda(\omega^2t^2(1-\lambda\omega)^{-2} + \omega t(1-\lambda\omega)^{-1}) + t\}^{1/2}$$

and $K(t,\lambda,\omega)$ is uniformly bounded for $t\in[0,t_0]$ as $\lambda\to 0^+$.

THEOREM C. *Let $A\in\mathcal{A}(\omega)$, $\omega\geqslant 0$ with $R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$. Then for the semigroup $\{T_\lambda(t); t\geqslant 0\}$ in Lemma B, $\lim_{\lambda\to 0^+} T_\lambda(t)x$ exists for $x\in\overline{D(A)}$ and $t\geqslant 0$. If we define $T(t)x = \lim_{\lambda\to 0^+} T_\lambda(t)x$, then the family $\{T(t); t\geqslant 0\}$ is a semigroup of type ω on $\overline{D(A)}$ and for $0<h<h_0$ and $x\in D(A)$*

$$\| T(t+h)x - T(t)x \| \leqslant h e^{\omega t}(2|Ax|+1).$$

In this paper we consider relations of the convergence of $\{A_n\}$ where $A_n\in\mathcal{A}(\omega_n)$, $\omega_n \geqslant 0$, the convergence of $\{J_{\lambda,n}\}$ where $J_{\lambda,n} = (I+\lambda A_n)^{-1}$ and the convergence of semigroups given by A_n in the sense of Theorem C in general Banach spaces. In particular we shall show that if functions A and A_n are closed m-accretive then $\lim_{n\to\infty} J_{\lambda,n} = J_\lambda$, $\lim_{n\to\infty} T_n(t) = T(t)$ and $\lim_{n\to\infty} A_n = A$ are equivalent when the dual X^* of X is uniformly convex using the following Theorem D where $\{T_n(t); t\geqslant 0\}$ and $\{T(t); t\geqslant 0\}$ are semigroups given by A_n and A in the sense of Theorem C respectively.

THEOREM D. *Let A and B be closed m-accretive subsets of $X\times X$ and let $\{T(t); t\geqslant 0\}$ and $\{S(t); t\geqslant 0\}$ be semigroups given by A and B in the sense of Theorem C respectively. Then $T(t)=S(t)$ for all $t\geqslant 0$ implies $D(A)=D(B)$ and $A=B$ when the dual X^* of X is uniormly convex.*

Proof. Since X^* is uniformly convex, X is reflexive together with X^*, therefore the Lipschitz continuous X-valued function $T(t)x$ with $x\in D(A)$ in $t\geqslant 0$ is strongly differentiable at a.e. t in $[0,\infty)$ (see Appendix in [8]). With the closedness and m-accretivity of A, $T(t)x$ is a unique solution of the Cauchy problem

$$\begin{cases} du(t)/dt + Au(t) \ni 0 \text{ a.e. t in } [0,\infty) \\ u(0)x = x \end{cases}$$

for $x\in D(A)$ (see Theorem II in [1]). Also $S(t)x$ is a unique solution of the Cauchy problem

$$\begin{cases} du(t)/dt + Bu(t) \ni 0 \text{ a.e. t in } [0,\infty) \\ u(0)x = x \end{cases}$$

for $x\in D(B)$. Hence Corollary 2 in [2] completes the proof.

Now we consider the main theorems.

THEOREM 1. *Let $A\in\mathcal{A}(\omega)$, $0\leqslant\omega\leqslant\alpha$ with $R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$ and let $A_n\in\mathcal{A}(\omega_n)$, $0\leqslant\omega_n\leqslant\alpha$, $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}$ for $0<\lambda\omega_n<1/2$. If*

(6) $$\lim_{n\to\infty} J_{\lambda,n}x=J_\lambda x \text{ for } x\in\overline{\text{co}D(A)}$$

and $D(A)\subset D(A_n)$ *for every* n, *then for every* $x\in\overline{D(A)}$

$$\lim_{n\to\infty} T_n(t)x=T(t)x$$

uniformly for t *in every bounded interval of* $[0,\infty)$ *where* $\{T_n(t);t\geq 0\}$ *and* $\{T(t);t\geq 0\}$ *are semigroups given by* A_n *and* A *in the sense of Theorem C respectively.*

Proof. By (6) $\lim_{n\to\infty} A_{\lambda,n}x=A_\lambda x$ for $x\in\overline{\text{co}D(A)}$, and hence there exists $M>0$ such that

(7) $$\|A_{\lambda,n}x\|\leqslant M \text{ and } \|A_\lambda x\|\leqslant M.$$

Choosing the integer m such that $t=m\lambda+h$ and $0\leqslant h<\lambda$, we have the estimate for $x\in\overline{D(A)}$

(8) $$\|T(t)x-T_n(t)x\|\leqq\|T(t)x-J^m_{\lambda,n}x\|+\|J^m_{\lambda,n}x-T_n(t)\|.$$

We estimate the first term of the left side in (8) as follows.

$$\|T(t)x-J^m_{\lambda,n}x\|\leqslant\|T(t)x-T_\lambda(t)x\|+\|T_\lambda(t)x-T_\lambda(m\lambda)x\| + \|T_\lambda(m\lambda)x-J_\lambda^m x\|+\|J_\lambda^m x-J^m_{\lambda,n}x\|$$

By (4), (5) and (7) we have

$$\|T_\lambda(t)x-T_\lambda(m\lambda)x\|\leqslant he^{\omega t/(1-\lambda\omega)}(\|A_\lambda x\|+1)\leqslant he^{\omega t/(1-\lambda\omega)}(M+1)$$

and

$$\|T_\lambda(m\lambda)x-J_\lambda^m x\|\leqslant\sqrt{\lambda}K(t,\lambda,\omega)\|A_\lambda x\|\leqslant\sqrt{\lambda}K(t,\lambda,\omega)M.$$

Thus we obtain

(9) $$\|T(t)x-J^m_{\lambda,n}x\| \leqslant\|T(t)x-T_\lambda(t)x\|+\lambda e^{\alpha t_0/(1-\lambda\alpha)}(M+1)+\sqrt{\lambda}K(t_0,\lambda,\alpha)M+\|J^m_\lambda x-J^m_{\lambda,n}x\|$$

for $t\in[0,t_0]$. We estimate the second term of the left side in (8) as follows.

$$\begin{aligned}&\|J^m_{\lambda,n}x-T_n(t)x\|\\&\leqslant\|J^m_{\lambda,n}x-J^m_{\lambda,n}J_{\lambda,n}x\|+\|J^m_{\lambda,n}J_{\lambda,n}x-T_{\lambda,n}(m\lambda)J_{\lambda,n}x\|\\&+\|T_{\lambda,n}(m\lambda)J_{\lambda,n}x-T_{\lambda,n}(t)J_{\lambda,n}x\|+\|T_{\lambda,n}(t)J_{\lambda,n}x-T_{\lambda,n}(t)x\|\\&+\|T_{\lambda,n}(t)x-T_n(t)x\|.\end{aligned}$$

Using Lemma A, (1), (5) and (7) we get

$$\|J^m_{\lambda,n}x-J^m_{\lambda,n}J_{\lambda,n}x\|\leqslant\lambda(1-\lambda\omega_n)^{-m}\|A_{\lambda,n}x\|\leqslant\lambda(1-\lambda\omega_n)^{-m}M,$$

$$\begin{aligned}\|J^m_{\lambda,n}J_{\lambda,n}x-T_{\lambda,n}(m\lambda)J_{\lambda,n}x\|&\leqslant\sqrt{\lambda}(1-\lambda\omega_n)^{-1}K(t,\lambda,\omega_n)|A_nJ_{\lambda,n}x|\\&\leqslant\sqrt{\lambda}(1-\lambda\omega_n)^{-1}K(t,\lambda,\omega_n)\|A_{\lambda,n}x\|\\&\leqslant\sqrt{\lambda}(1-\lambda\omega_n)^{-1}K(t,\lambda,\omega_n)M,\end{aligned}$$

$$\begin{aligned}\|T_{\lambda,n}(m\lambda)J_{\lambda,n}x-T_{\lambda,n}(t)J_{\lambda,n}x\|&\leqslant\delta e^{\omega_n t/(1-\lambda\omega_n)}(2|A_nJ_{\lambda,n}x|+1)\\&\leqslant\lambda e^{\omega_n t/(1-\lambda\omega_n)}(2M+1)\end{aligned}$$

and

$$\|T_{\lambda,n}(t)J_{\lambda,n}x-T_{\lambda,n}(t)x\|\leqslant\lambda e^{\omega_n t/(1-\lambda\omega_n)}\|A_{\lambda,n}x\|\leqslant\lambda e^{\omega_n t/(1-\lambda\omega_n)}M.$$

Thus we have

(10) $$\|J^m_{\lambda,n}x-T_n(t)x\|\leqslant\lambda(1-\lambda\alpha)^{-m}M+\sqrt{\lambda}K(t_0,\lambda,\alpha)M$$

$$+\lambda e^{\alpha t_0/(1-\lambda\alpha)}(2M+1)+\lambda e^{\alpha t_0/(1-\lambda\alpha)}M+\|T_{\lambda,n}(t)x-T_n(t)x\|$$

for $t\in[0,t_0]$. It follows from (9) and (10) that

$$\begin{aligned}&\|T(t)x-T_n(t)x\|\\&\leqslant\|T(t)x-T_\lambda(t)x\|+\{\lambda(1-\lambda\alpha)^{-m}+2\sqrt{\lambda}K(t_0,\lambda,\alpha)+\lambda e^{\alpha t_0/(1-\lambda\alpha)}\}M\\&\quad+\|J_\lambda^m x-J_{\lambda,n}^m x\|+2\lambda e^{\alpha t_0/(1-\lambda\alpha)}(2M+1)+\|T_{\lambda,n}(t)x-T_n(t)x\|\end{aligned}$$

for $x\in\overline{D(A)}$ and $t\in[0,t_0]$. First for each $\varepsilon>0$ we fix $\lambda>0$ sufficiently small such that

$$\|T(t)x-T_\lambda(t)x\|<\varepsilon/5,$$

$$\{\lambda(1-\lambda\alpha)^{-m}+2\sqrt{\lambda}K(t_0,\lambda,\alpha)+\lambda e^{\alpha t_0/(1-\lambda\alpha)}\}M<\varepsilon/5,$$

$$2\lambda e^{\alpha t_0/(1-\lambda\alpha)}(2M+1)<\varepsilon/5$$

and

$$\|T_{\lambda,n}(t)x-T_n(t)x\|<\varepsilon/5$$

for every n. Next we choose n sufficiently large such that

$$\|J_\lambda^m x-J_{\lambda,n}^m x\|<\varepsilon/5.$$

Thus it follows that for $x\in\overline{D(A)}$

$$\lim_{n\to\infty}T_n(t)x=T(t)x$$

uniformly for t in every bounded interval of $[0,\infty)$.

THEOREM 2. *Let $A_n\in\mathcal{A}(\omega_n)$, $0\leqslant\omega_n\leqslant\alpha$, $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}$ for $0<\lambda\omega_n<1/2$. If $D\subset D(A_n)$ for every n and $\lim_{n\to\infty}J_{\lambda,n}x$ exists in D for $x\in\overline{\mathrm{co}D(A)}$ and some $\lambda>0$, we denote the limit by $J_\lambda x$, then there exists $A\in\mathcal{A}(0)$ such that $J_\lambda=(I+\lambda A)^{-1}$ and $R(I+\lambda A)\supset\overline{\mathrm{co}D}$ for $\lambda>0$. Moreover for every $x\in\overline{D(A)}$*

(11) $$\lim_{n\to\infty}T_n(t)x=T(t)x$$

uniformly for t in every bounded interval of $[0,\infty)$ where $\{T_n(t);t\geqslant0\}$ are semigroups given by A_n and A in the sense of Theorem C respectively.

Proof. The limit $\lim_{n\to\infty}J_{\lambda,n}x$ exists in D for all $\lambda>0$ (see [4]). If we define $A\subset X\times X$ by

$$\bigcup_{\lambda>0}\{[J_\lambda x,\ \lambda^{-1}(I-J_\lambda x)];\ x\in\overline{\mathrm{co}D}\}$$

then clearly $A\in\mathcal{A}(0)$. For $x\in\overline{\mathrm{co}D}$, from

(12) $$\lambda^{-1}(I-J_\lambda)x\in AJ_\lambda x$$

we have $x\in(I+\lambda A)J_\lambda x\subset R(I+\lambda A)$, that is,

(13) $$\overline{\mathrm{co}D}\subset R(I+\lambda A).$$

Also by (12) we have $J_\lambda x=(I+\lambda A)^{-1}x$ for $x\in\overline{\mathrm{co}D}$. For $y\in D(A)$, there exists $x\in\overline{\mathrm{co}D}$ such that $y=J_\lambda x\in D$, and hence $D(A)\subset D$. By (13) $\overline{\mathrm{co}D(A)}\subset R(I+\lambda A)$ for $\lambda>0$. Therefore it follows from Theorem 1 that (11) holds true.

THEOREM 3. *Let A be a function in the class $\mathcal{A}(\omega)$, $0\leqslant\omega\leqslant\alpha$ with $R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$ and let A_n be a function in the class $\mathcal{A}(\omega_n)$, $0\leqslant\omega_n\leqslant\alpha$ such that $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}$ for $0<\lambda\omega_n<1/2$. If $D(A)\subset D(A_n)$ for every n and $\lim_{n\to\infty}A_nx=Ax$ for*

$x\in D(A)$ then for every $x\in\overline{D(A)}$

$$\lim_{n\to\infty} T_n(t)x=T(t)x \tag{14}$$

uniformly for t in every bounded interval of $[0, \infty)$ where $\{T_n(t); t\geqslant 0\}$ and $\{T(t); t\geqslant 0\}$ are semigroups given by A_n and A in the sense of Theorem C respectively.

Proof. For $x\in\overline{\mathrm{co}D(A)}$ there exists $y\in D(A)$ such that $x=y+\lambda Ay$ and $\lim_{n\to\infty} A_ny=Ay$. We have the estimate

$$\begin{aligned}\| J_{\lambda,n}x-J_\lambda x \| &= \| J_{\lambda,n}x-y \| = \| J_{\lambda,n}x-J_{\lambda,n}(y+\lambda A_ny) \| \\ &\leqslant (1-\lambda\omega_n)^{-1} \| x-(y+\lambda A_ny) \| \\ &= (1-\lambda\omega_n)^{-1} \| y+\lambda Ay-y-\lambda A_ny \| \\ &\leqslant (1-\lambda\alpha)^{-1} \| Ay-A_ny \| .\end{aligned} \tag{15}$$

Thus $\lim_{n\to\infty} J_{\lambda,n}x=J_\lambda x$ for $x\in\overline{\mathrm{co}D(A)}$. From Theorem 1, (14) holds true.

COROLLARY 4. *Let A_n be a function in the class $\mathcal{A}(\omega_n)$, $\omega_n\geqslant 0$ such that $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}$ for $0<\lambda\omega_n<1/2$. Supposing that $\lim_{n\to\infty}\omega_n$ exists and $\lim_{n\to\infty} A_nx$ exists for $x\in E$ $(\subset\bigcap_{n=1}^{\infty}D(A_n))$ we denote the limits by ω and Ax for $x\in E$ respectively. If $D(A)=D(A_n)$ for every n then $A\subset\mathcal{A}(\omega)$, $\omega\geqslant 0$ and $R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$. Moreover for every $x\in\overline{D(A)}$*

$$\lim_{n\to\infty} T_n(t)x=T(t)x \tag{16}$$

uniformly for t in every bounded interval of $[0,\infty)$, where $\{T_n(t); t\geqslant 0\}$ and $\{T(t); t\geqslant 0\}$ are semigroups given by A_n and A in the sense of Theorem C respectively.

Proof. Since $A_n\in\mathcal{A}(\omega_n)$, for x_1, $x_2\in D(A)$ we have

$$\| (x_1+\lambda A_nx_1)-(x_2+\lambda A_nx_2) \| \geqslant (1-\lambda\omega_n) \| x_1-x_2 \| .$$

As $n\to\infty$

$$\| (x_1+\lambda Ax_1)-(x_2+\lambda Ax_2) \| \geqslant (1-\lambda\omega) \| x_1-x_2 \| .$$

Thus $A\in\mathcal{A}(\omega)$, $\omega\geqslant 0$. Since $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}=\overline{\mathrm{co}D(A)}$, for $z\in\overline{\mathrm{co}D(A)}$ there exists $x\in D(A_n)$ such that $z=x+\lambda A_nx$. As $n\to\infty$ there exists $x\in D(A)$ such that $z=x+\lambda Ax\in R(I+\lambda A)$, that is, $R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$. Therfore by Theorem 3, (16) holds true.

THEOREM 5. *Let A be a function in the class $\mathcal{A}(\omega)$, $0\leqslant\omega\leqslant\alpha$ with $\overline{R(I+\lambda A)}\supset\overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$. Let A_n be a function in the class $\mathcal{A}(\omega_n)$, $0\leqslant\omega_n\leqslant\alpha$ for each n such that $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}$ for $0<\lambda\omega_n<1/2$. If $\lim_{n\to\infty} A_nx=Ax$ for $x\in D(A)$ and $D(\bar{A})\subset D(A_n)$ for every n, then $\bar{A}\in\mathcal{A}(\omega)$, $0\leqslant\omega\leqslant\alpha$ with $R(I+\lambda A)\supset\overline{\mathrm{co}D(\bar{A})}$ for $0<\lambda\omega<1/2$. Moreover for $x\in\overline{D(A)}=\overline{D(\bar{A})}$*

$$\lim_{n\to\infty} T_n(t)x=T(t)x \tag{17}$$

uniformly for t in every bounded interval of $[0,\infty)$ where $\{T_n(t); t\geqslant 0\}$ and $\{T(t); t\geqslant 0\}$ are semigroups given by A_n and $\bar{A}$ in the sense of Theorem C respectively.

Proof. For $z\in\overline{\mathrm{co}D(A)}\subset\overline{R(I+\lambda A)}$ there exists $z_n\in R(I+\lambda A)$ and $x_n\in D(A_n)$ such that

$$z_n=x_n+\lambda Ax_n \text{ and } \lim_{n\to\infty} z_n=z. \tag{18}$$

Since $A\in\mathcal{A}(\omega)$, we have for x_n, $x_m\in D(A)$

$$\|(x_n+\lambda Ax_n)-(x_m+\lambda Ax_m)\|\geq(1-\lambda\omega)\|x_n-x_m\|,$$

and hence $\lim_{n\to\infty} x_n$ exists, we denote the limit by x. By (18) $\lim_{n\to\infty} Ax_n$ exists and it equals to $\bar{A}x$ for $x\in D(\bar{A})$. As $n\to\infty$ we have from (18) $z=x+\lambda\bar{A}x\in R(I+\lambda\bar{A})$. Thus $\overline{\mathrm{co}D(A)}\subset R(I+\lambda\bar{A})$. Since $\overline{\mathrm{co}D(\bar{A})}=\overline{\mathrm{co}D(A)}$ we obtain $R(I+\lambda\bar{A})\supset\overline{\mathrm{co}D(\bar{A})}$ for $0<\lambda\omega<1/2$. Since $A\in\mathcal{A}(\omega)$, we get for $x_{1,n}$, $x_{2,n}\in D(A)$ such that $\lim_{n\to\infty} x_{1,n}=x_1$ and $\lim_{n\to\infty} x_{2,n}=x_2$,

$$\|(x_{1,n}+\lambda Ax_{1,n})-(x_{2,n}+\lambda Ax_{2,n})\|\geqslant(1-\lambda\omega)\|x_{1,n}-x_{2,n}\|.$$

As $n\to\infty$

$$\|(x_1+\lambda\bar{A}x_1)-(x_2+\lambda\bar{A}x_2)\|\geqslant(1-\lambda\omega)\|x_1-x_2\|$$

for $x_1, x_2\in D(\bar{A})$, and hence $\bar{A}\in\mathcal{A}(\omega)$. Set $\bar{J}_\lambda=(I+\lambda\bar{A})^{-1}$ and $\bar{A}_\lambda=\lambda^{-1}(I-\bar{J}_\lambda)$. Since $\bar{A}_\lambda x\in\bar{A}\bar{J}_\lambda x$, there exists $y_m\in D(A)$ such that $\lim_{m\to\infty} y_m=\bar{J}_\lambda x$ and $\lim_{m\to\infty} Ay_m=\bar{A}_\lambda x$. Therefore we obtain $\lim_{m\to\infty}(y_m+\lambda Ay_m)=x$. We have the estimate

$$\|J_{\lambda,n}x-\bar{J}_\lambda x\|\geqslant\|J_{\lambda,n}x-y_m\|+\|y_m-\bar{J}_\lambda x\|$$

for $x\in\overline{\mathrm{co}D(A)}$. Since $A_n\in\mathcal{A}(\omega_n)$, $0\leqslant\omega_n\leqslant\alpha$,

$$\begin{aligned}\|J_{\lambda,n}x-y_m\|&\leqslant(1-\lambda\omega_n)^{-1}\|(J_{\lambda,n}x+\lambda A_nJ_{\lambda,n}x)-(y_m+\lambda A_ny_m)\|\\&\leqslant(1-\lambda\alpha)^{-1}\|x-(y_m+\lambda A_ny_m)\|\\&\leqslant(1-\lambda\alpha)^{-1}\{\|x-(y_m+\lambda Ay_m)\|+\lambda\|Ay_m-A_ny_m\|\}.\end{aligned}$$

First, for every $\varepsilon>0$, we fix m sufficiently large such that

$$(1-\lambda\alpha)^{-1}\|x-(y_m+\lambda Ay_m)\|<\varepsilon/3,\quad \|y_m-\bar{J}_\lambda x\|<\varepsilon/3.$$

Next we choose n sufficiently large such that

$$\lambda(1-\lambda\alpha)^{-1}\|Ay_m-A_ny_m\|<\varepsilon/3.$$

Thus it follows that for $x\in\overline{\mathrm{co}D(A)}$, $\lim_{n\to\infty} J_{\lambda,n}x=\bar{J}_\lambda x$. Hence by Theorem 1, (17) holds true.

COROLLARY 6. *Let A_n be a function in the class $\mathcal{A}(\omega_n)$, $0\leqslant\omega_n\leqslant\alpha$ such that $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}$ for $0<\lambda\omega_n<1/2$ and $\lim_{n\to\infty}\omega_n=\omega$ exists. If $\lim_{n\to\infty} A_nx$ exists for $x\in E\,(\subset\bigcap_{n=1}^{\infty} D(A_n))$ we denote the limit by Ax for $x\in E$, and if $D(A)=D(A_n)$ for every n, then A and $\bar{A}$ are in the class $\mathcal{A}(\omega)$, $0\leqslant\omega\leqslant\alpha$ with $R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ and $R(I+\lambda\bar{A})\supset\overline{\mathrm{co}D(\bar{A})}$ for $0<\lambda\omega<1/2$. Moreover for every $x\in\overline{D(A)}$*

$$\lim_{n\to\infty} T_n(t)x=T(t)x$$

uniformly for t in every bounded interval of $[0,\infty)$ where $\{T_n(t);t\geqslant0\}$ and $\{T(t);t\geq0\}$ are semigroups given by A_n and $\bar{A}$ in the sense of Theorem C respectively.

Proof. Since $A_n\in\mathcal{A}(\omega_n)$ we obtain for x_1, $x_2\in D(A)=D(A_n)$

$$\|(x_1+\lambda A_nx_1)-(x_2+\lambda A_nx_2)\|\geqslant(1-\lambda\omega_n)\|x_1-x_2\|.$$

As $n\to\infty$ we have

$$\|(x_1+\lambda Ax_1)-(x_2+\lambda Ax_2)\|\geq(1-\lambda\omega)\|x_1-x_2\|.$$

Thus $A\in\mathcal{A}(\omega)$, $0\leqslant\omega\leqslant\alpha$. From $R(I+\lambda A_n)\supset\overline{\mathrm{co}D(A_n)}=\overline{\mathrm{co}D(A)}$, for $z\in\overline{\mathrm{co}D(A)}$ there exists $x\in D(A)=D(A_n)$ such that $z=x+\lambda A_nx$. As $n\to\infty$ we have $z=x+\lambda Ax\in R(I+\lambda A)$. Thus $R(\overline{I+\lambda A})\supset R(I+\lambda A)\supset\overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$, and hence $\bar{A}\in\mathcal{A}(\omega)$, $0\leqslant\omega\leqslant\alpha$. Therefore by Theorem 5 the proof is complete.

As an application of Corollary 4 we consider the following.

THEOREM 7. *Let S_n be an operator from a closed convex subset X_0 of X into itself such that*

$$\| S_n x - S_n y \| \leqslant e^{\omega h_n} \| x-y \| \quad \text{for } x, y \in X_0$$

where $\omega \geqslant 0$ and $h_n > 0$, $h_n \to 0^+$ as $n \to \infty$. Suppose that $\lim_{n\to\infty} h_n^{-1}(I-S_n)x$ exists for every $x \in X_0$, we denote the limit by Ax. If $D(\bar{A}) \subset X_0$ and $\overline{R(I+\lambda A)} \supset \overline{\mathrm{co}D(A)}$ for $0<\lambda\omega<1/2$, then $\bar{A} \in \mathcal{A}(\omega)$ *and for every $x \in \overline{D(A)}$,*

$$T(t)x = \lim_{n\to\infty} \left(I + \frac{t}{n}\bar{A}\right)^{-n} x$$

uniformly for t in every bounded interval of $[0, \infty)$ where $\{T(t); t \geqslant 0\}$ is the semigroup given by $\bar{A}$ in the sense of Theorem C.

Proof. Put $A_n = h_n^{-1}(I-S_n)$, then $D(A_n) = X_0$. For $x_1, x_2 \in D(A)$,

$$\begin{aligned} &\| (x_1 + \lambda A_n x_1) - (x_2 + \lambda A_n x_2) \| \\ &\quad \geqslant \| (1+\lambda h_n^{-1})(x_1 - x_2) \| - \lambda h_n^{-1} \| S_n x_1 - S_n x_2 \| \\ &\quad \geqslant \{1 - \lambda h_n^{-1}(e^{\omega h_n} - 1)\} \| x_1 - x_2 \| . \end{aligned}$$

Set $\omega_n = h_n^{-1}(e^{\omega h_n} - 1)$. Then

(19) $$\| (x_1 + \lambda A_n x_1) - (x_2 + \lambda A_n x_2) \| \geqslant (1 - \lambda\omega_n) \| x_1 - x_2 \| ,$$

and hence $A_n \in \mathcal{A}(\omega_n)$ and $\lim_{n\to\infty} \omega_n = \omega$. As $n \to \infty$ in (19) we have

$$\| (x_1 + \lambda A x_1) - (x_2 + \lambda A x_2) \| \geqslant (1 - \lambda\omega) \| x_1 - x_2 \| .$$

Thus $A \in \mathcal{A}(\omega)$, and since $\overline{R(I+\lambda A)} \supset \overline{\mathrm{co}D(A)}$ for $0 < \lambda\omega < 1/2$, we obtain $\bar{A} \in \mathcal{A}(\omega)$ and $R(I+\lambda\bar{A}) \supset \overline{\mathrm{co}D(A)} = \overline{\mathrm{co}D(\bar{A})}$ as in the proof of Theorem 5. We shall show that $R(I+\lambda\bar{A}_n) \supset \overline{\mathrm{co}D(A_n)} = X_0$ for sufficiently small $\lambda > 0$. Let $z \in X_0$. We define a mapping K from X_0 into itself by

(20) $$Kx = (1+\lambda/h_n)^{-1} z + \lambda h_n^{-1} (1+\lambda/h_n)^{-1} S_n x.$$

For $x_1, x_2 \in X_0$, we have

$$| Kx_1 - Kx_2 | \leqslant \lambda h_n^{-1} e^{\omega h_n} (1+\lambda/h_n)^{-1} \| x_1 - x_2 \| ,$$

that is, K is a strict contraction for sufficiently small $\lambda > 0$. Hence there exists $x \in X_0$ such that $Kx = x$. By (20)

$$\begin{aligned} x &= (1+\lambda/h_n)^{-1} z + \lambda h_n^{-1} (1+\lambda/h_n)^{-1} S_n x, \\ z &= x + \lambda A_n x \in R(I+\lambda A_n) . \end{aligned}$$

Thus $R(I+\lambda A_n) \supset \overline{\mathrm{co}D(A_n)}$ for sufficiently small $\lambda > 0$. It follows from Theorem 5 that for every $x \in \overline{D(A)}$

$$\lim_{n\to\infty} T_n(t)x = T(t)x$$

uniformly for t in every bounded interval of $[0, \infty)$, where $\{T_n(t); t \geqslant 0\}$ is a semigroup given by A_n in the sense of Theorem C. For every $x \in D(A) = D(\bar{A})$ we have the estimate

$$\begin{aligned} \| T(t)x - \bar{J}^n_{t/n} x \| \leqslant &\| T(t)x - T_m(t)x \| + \| T_m(t)x - T_{t/n,m}(t)x \| \\ &+ \| T_{t/n,m}(t)x - J^n_{t/n,m} x \| + \| J^n_{t/n,m} x - \bar{J}^n_{t/n} x \| . \end{aligned}$$

By (5) we have

$$\| T_{t/n,m}(t)x - J^n_{t/n,m} x \| \leqslant \sqrt{t/n}\, K(t, t/n, \omega) \| A_{t/n,m} x \| \leqslant \sqrt{t_0/n}\, K(t_0, t_0/n, \omega) M.$$

Let $t\in[0,t_0]$. First, for every $\varepsilon>0$ we fix m sufficiently large such that

$$\| T(t)x-T_m(t)x \| <\varepsilon/4$$

and

$$\| J^n_{t/n,m}x-\bar{J}^n_{t/n}x \| <\varepsilon/4$$

for every n. Next, we choose n sufficiently large such that

$$\| T_m(t)x-T_{t/n,m}x \| <\varepsilon/4$$

and

$$\sqrt{t_0/n}\ K(t_0,\ t_0/n,\omega)M<\varepsilon/4.$$

Accordingly for every $x\in\overline{D(A)}$

$$T(t)x=\lim_{n\to\infty}\ (I+t/n\bar{A})^{-n}$$

uniformly for t in every bounded interval of $[0,\ \infty)$.

THEOREM 8. *Let X^* be uniformly convex and let A_n and A be closed m-accretive functions. If $D(A_n)=D(A)$,*

(21) $$\lim_{n\to\infty}\ T_n(t)x=T(t)x$$

for $x\in\overline{D(A)}$ and $\lim_{n\to\infty} A_nx$ exists for $x\in D(A)$ then $\lim_{n\to\infty} A_nx=Ax$ for $x\in D(A)$, where $\{T_n(t);\ t\geqslant0\}$ and $\{T(t);\ t\geqslant0\}$ are semigroups given by A_n and A in the sense of Theorem C respectively.

Proof. Put $\lim_{n\to\infty} A_nx=Bx$ for $x\in D(B)=D(A)$, then B is closed m-accretive. Let $\{S(t);\ t\geqslant0\}$ be a semigroup given by B in the sense of Theorem C. By Corollary 4 for every $x\in\overline{D(A)}$

(22) $$\lim_{n\to\infty}\ T_n(t)x=S(t)x$$

uniformly for t in every bounded interval of $[0,\ \infty)$. By (21) and (22) we have for $x\in\overline{D(A)}=\overline{D(B)}$, $T(t)x=S(t)x$. It follows from Theorem D that $Ax=Bx$ for $x\in D(A)=D(B)$. Hence we obtain $\lim_{n\to\infty} A_nx=Ax$ for $x\in D(A)$.

REMARK. Let X be a Banach space the dual of which is uniformly convex and let A and A_n be closed m-accretive functions. Put $J_\lambda=(I+\lambda A)^{-1}$ and $J_{\lambda,n}=(I+\lambda A_n)^{-n}$. Suppose that $\{T_n(t);\ t\geqslant0\}$ and $\{T(t);\ t\geqslant0\}$ are semigroups given by A_n and A in the sense of Theorem C respectively. By Theorem 1, Theorem 8 and (15), the following 1), 2) and 3) are equivalent:

1) $\lim_{n\to\infty} J_{\lambda,n}=J_\lambda$,

2) $\lim_{n\to\infty} T_n(t)=T(t)$ uniformly for $t\in[0,t_0]$,

3) $\lim_{n\to\infty} A_n=A$.

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