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ON THE GROSS DOMAIN OF A MEROMORPHIC FUNCTION

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Pommerenke and McMillan in [1] have defined a Gross domain of analytic functions in the unit disc. In this paper we define a star-shaped domain on the Riemann sphere and a Gross domain of a meromorpic function in the unit disc. Using this definition we prove that if $f(z)$ is a meromorphic function in the unit disc. without Koebe arcs, which has asymptotic values on a dense set in an arc α on the unit circle, then for each point ζ on the arc α, either $f(z)$ has an asymptotic valve at ζ, or every neighborhood of ζ contains non-degenerate Gross domains of $f(z)$.

Let S be a domain on the Riemann sphere Ω, and let P be a point in S, Q a point in Ω. By a suitable linear transformation L the point P may be transformed to the south pole, the point Q to the north pole. Then S is said to be (P, O)*-star-shaped* if the stereographic projection of $L(S)$ is star-shaped with respect to the origin in the complex plane.

A domain S on the Riemann sphere is said to be star-shaped if it is (P, O)-star-shaped for some point P in S (the point P is called a center of S) and Q in Ω.

A *Gross domain* G of a meromorphic function on D is defined to be a subdomain of D having the following properties:

(a) $f(z)$ maps G one-to-one onto a star-shaped domain S on the Riemann sphere and

(b) G is not properly contained in any other subdomain of D having the property (a).

The point in D corresponding to the center of S is said to be the *center* of the Gross domain G.

The arcs in D corresponding to the "rays" of S are defined to be the *rays* of G.

At most countably many rays of G join the center of G to the points in D which corresponed to branch points of the Riemann Surface onto which f maps the unit disc. Any other ray of G joins the center of G to some point ζ on the unit circle, because otherwise $f(z)$ would have Koebe arcs: and $f(z)$ has an asymptotic valve at ζ (which is "rectilinearly" accessible on the Riemann sphere). We note that if $f(z)$ is normal its asymptotic values are angular limits, by the theorem of Lehto and Virtanen [1].

It is clear that every point in D which does not correspond to a branch point of the Riemann surface (onto which f maps the unit disc) is the center of a Gross domain of $f(z)$.

We begin by showing a lemma which will be used in the the proof of Theorem 1.

LEMMA 1. *Let $f(z)$ be a meromorphic function on the unit disc, without Koebe arcs,*

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and assume that $f(z)$ has asymptotic values on a dense set on the unit circle C, and let J_0 be an open arc on C. Assume that there exists a sequence of analytic Jordan arcs $J_n \subset D$ such that $J_n \to J_0$ and $f(z)$ maps each J_n one-to-one onto a circle arc on the Riemann sphere. Then at each point ζ on J_0, $f(z)$ has an asymptotic value (which is "rectilinearly accessible" on the Riemann sphere.)

Proof. By taking a suitable subarc of J_n and by choosing a subsequence of J_n, if necessary, we may assume that the spherical length of each $f(J_n)$ is not greater than $\frac{\pi}{2}$ and that J_n converges to a subarc of J_0 containing ζ. Without loss of generality we may assume that the end points of the "segments" (circle arcs) $f(J_0)$ converge, respectively to the points w' and w'' on the Riemann sphere.

We may also assume that the "directions" of the segments $f(J_0)$ converge, and consequently that $f(J_n)$ "converges" as $n \to \infty$ to a "segment" L joining w' and w'' (which may be a single point if $w' = w''$). Choose a sequence $\{\zeta_m\}$ on C, converging to ζ, so that $f(z)$ has asymptotic values $f(\zeta_m)$ along the Jordan arcs β_m at ζ_m, for each m. By choosing a subsequence of $\{\zeta_m\}$, if necessary, we may assume that $f(\zeta_m)$ converges to w_0 "monotonically". Choose n_i and m_i (i=1, 2, ...) so that

(1) J_{n_i} intersect β_{m_i} at the last point of intersection $z(n_i, m_i)$, and

(2) J_{n_i} intersect β_{m_i} at $z(n_{i+1}, m_i)$, $\beta_{m_{i+1}}$ at the last point of intersection $z(n_{i+1}, m_{i+1})$.

Let γ_i be the subarc of β_{m_i} between $z(n_i, m_i)$ and $z(n_{i+1}, m_i)$, and let γ'_i be the subarc of $J_{n_{i+1}}$ between $z(n_{i+1}, m_i)$ and $z(n_{i+1}, m_{i+1})$.

We choose n_i and $m_i (i=1, 2, \dots)$ so that the spherical length of $f(\gamma'_i) \to 0$ as $i \to \infty$. Consider

$$\gamma = \gamma_1 + \gamma_1' + \gamma_2 + \gamma_2' + \cdots .$$

Then along this arc at ζ, $f(z)$ has the asymptotic value w_0. This completes the proof.

THEOREM 1. *Let $f(z)$ be a meromorphic function in the unit disc, without Koebe arcs, which has asymptotic values on a dense set in an arc α on the unit circle C. Then for each point ζ on the arc α, either,*

(a) *$f(z)$ has an asymptotic value at ζ, or*

(b) *every neighborhood of ζ contains non-degenerate Gross domains of $f(z)$, and furthermore $\delta(\zeta, d) \to 0$ as $d \to 0$, where $\delta(\zeta, d)$ denotes the supremum of the euclidean diameters of the Gross domains of $f(z)$ intersecting $\{z: |z-\zeta| < d\}$.*

Proof. We assume that $\limsup_{d \to 0} \delta(\zeta, d) > 0$ and prove that $f(z)$ has an asymptotic value at ζ. By this assumption there exists a sequence of Jordan arcs $J_{1,n} \subset D$ having the following properties:

(a) $f(z)$ maps each $J_{1,n}$ one-to-one onto a segment (of a circle) $f(J_{1,n})$ on the Riemann sphere;

(b) for some r_1 $(0 < r_1 < 1)$ every $J_{1,n}$ has an end point on $\{z: |z-\zeta| = r_1\}$ and lies, except for this point, in $\{z: |z-\zeta| < r_1\}$; and

(c) the other endpoint $z_{1,n}$ tends to ζ.

If there exists an arc J_0 on C having ζ as an endpoint and satisfying the hypothesis of the lemma, clearly $f(z)$ has an asymptotic value at ζ. Thus we only need to consider the case where no such J_0 exists. Then by the lemma there exists some point $z_1 \in D$, $|z_1 - \zeta| < r_1$, such that every neighborhood of z_1 intersects infinitely many $J_{1,n}$. Set $r_2 = \frac{|z_1 - \zeta|}{2}$, and let $\{J_{1,n_k}\}$ be a subsequence of $\{J_{1,n}\}$ such that every neighborhood of z_1 intersects all except finitely many J_{1,n_k} and such that every z_{1,n_k} is in $\{z; |z-\zeta| < r_2\}$. Let $J_{2,k}$ be the subarc of J_{1,n_k} joining J_{1,n_k} to a point on $\{z: |z-\zeta| = r_2\}$ and lying, except for this point, in $\{z: |z-\zeta| < r_2\}$. Again by the lemma there exists some point $z_2 \in D \cap \{z: |z-\zeta| < r_2\}$, such that every neighborhood of z_2 intersects infinitely many $J_{2,k}$. Continuing in this way, we define a subsequence $\{J_{m,m}\}$. Since the $f(J_{m,n})$ are segments of circles, all of the points $w_m = f(z_m)$ lie on the same segment of a circle. They are all distinct because otherwise $f(z)$ would be constant. Furthermore the w_m tend "monotonically" along this segment to a finite or infinite w_0. Hence

$$L = [w_1, \ w_2] + [w_2, \ w_3] + \cdots$$

is a segment of a circle ending at w_0.

Let D_m be an open disc about z_m contained in $\{z: |z - z_m| < r_m\}$ such that the spherical diameter of $f(D_m)$ tends to zero as m tends to ∞. For each m there exists an n such that $J_{m,n}$ contains a subarc J_m having its initial point in D_m and its terminal point in D_{m+1}. We join the terminal point of J_m to the initial point of J_{m+1} by an arc L_m lying in D_{m+1}. Then

$$\Gamma = J_1 + L_1 + J_2 + L_2 + \cdots$$

is an arc in D ending at ζ along which $f(z)$ has the asymptotic value w_0.

We now prove that every neighborhood of ζ contains non-degenerate Gross domains of $f(z)$, provided that $\delta(\zeta, d) \to 0$ as $d \to 0$. We only need to consider the case where every neighborhood of ζ contains a Gross domain whose image under $f(z)$ is the Riemann sphere minus a closed circle arc.

Consider any Gross domain G with this property, and take the closed circle are to be the closed great circle arc whose stereographic projection is the non-negative real axis. We now think of $f(G)$ as being on the Riemann surface R over the Riemann sphere onto which $f(z)$ maps D. There is a largest subdomain R_0 of R containing $f(G)$ and having the property that R_0 is a copy of the Riemann surface over the Riemann sphere onto which e^z maps a domain of the form

$$\{x+iy: -\infty \leqslant y_1 < y_2 \leqslant \infty\}.$$

Either $y_1 > -\infty$ or $y_2 < +\infty$, because R is of hyperbolic type, and thus cannot contain a copy of the logarithmic surface, otherwise $f^{-1} \circ e^z$ would map the whole plane into D and f must be a constant. Thus there exists a point $P_0 \in R_0$ with the following properties: $|w_0|$ is large, where w_0 is the stereographic projection of the projection of P_0 onto the Riemann sphere; and P_0 is near an "edge" of R_0, in the sense that some curve on R_0 beginning at P_0 and tending to the boundary of R_0 is such that the euclidean diameter of the stereographic projection of the projection onto the Riemann sphere is small. Let Z_0 be the point of D corresponding under $f(z)$ to P_0. The Gross domain G_0 of $f(z)$ whose center is Z_0 has a ray whose image under $f(z)$ is a segment joining w_0 to the south pole.

If the south pole is an interior point of the ray, there exists a small disc about this point, and then R could be extended, contrary to the choice of R.

If $f(G_0)$ were the sphere minus a closed great circle arc connecting the south pole to the north pole, we could make R_0 larger, contrary to the definition of R_0. Thus G_0 is a non-degenerate gross domain of $f(z)$.

The point Z_0 can be joined to a point of G by a Jordan arc the stereographic projection of whose image under $f(z)$ lies on $\{w: |w|=|w_0|\}$. Thus, since $f(z)$ has no sequence of Koebe arcs for the value ∞, we can make G_0 (which depends on P_0) have points as near as we like to G by taking w_0 to be sufficiently large (for, let $C^{(n)}$ and $G_0^{(n)}$ be the corresponding arc, Gross domain, the nondegenerate gross domain, respectively, then since $f(z)$ has no Koebe arcs, the diameter of $C^{(n)}$ tends to 0 as $n \to \infty$. Hence the distance between $G^{(n)}$ and $G_0^{(n)}$ tends to zero as n tends to infinity). Therefore, since $\delta(\zeta, d) \to 0$, every neighforhood of ζ contains ternary Gross domains of $f(z)$. This completes the proof of Theorem 1.

REMARK: The main argument in the above proof is essentially found in Pommerenke and McMillan [1]. We have modified their proof using Lemma 1.

As a corollary we have the following

THEOREM 2. *Let $f(z)$ be a normal meromorphic function in the unit disc, which has asymptotic values on a dense set on the unit circle C. Then for each point on the unit circle C, either*

(a) *$f(z)$ has finite angular limits at almost all points of some open arc containing ζ, or*

(b) *every neighborhood of ζ contains non-degenerate Gross domains $f(z)$, and furthermore $\delta(\zeta, d) \to 0$ as $d \to 0$, where $\delta(\zeta, d)$ denotes the supremum of the euclidean diameters of the Gross domains of $f(z)$ intersecting*

$$\{z: |z-\zeta| < d\}.$$

Proof. Bagehmihl and Seidel [1] have shown that a non-constant normal meromorphic function has no Koebe arcs. Use Theorem 1 and the well known theorem of Lehto and Virtanen [1] that if a normal meromorphic function has the asymptotic value C at ζ, then $f(z)$ has the angular limit C at ζ. Then Lusin-Privalov theorem (Privalov [1], p. 212) that almost all angular limits are finite completes the proof.

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