ON MODULES OVER \((qa)\)-RINGS

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All rings considered will have a unit and modules will be unital left modules. In a recent paper [4], H. Harui has considered modules over commutative rings whose total quotient ring is Artinian, called \((qa)\)-rings. The purpose of this note is to extend two of the main results of his paper to non-commutative rings. Specifically, we show (Theorem 2.2) that for a ring \(R\) having a quasi-Frobenius quotient ring, if \(M\) is an \(h\)-divisible \(R\)-module such that \(M/l(M)\) is an injective \(R\)-module then \(l(M)\) is a direct summand of \(M\); this extends Theorem 2.10 of [4]. Moreover, extending Theorem 3.3 of [4], we show that if \(R\) is a ring with an Artinian quotient ring \(Q\), then \(Q\) is semisimple if and only if \(E(M)/M\) is a torsion \(R\)-module for all \(R\)-modules \(M\).

1. Preliminaries.

Our notation will be the same as that of [4]. Thus let \(S\) denote the set of non-zero divisors of \(R\). An \(R\)-module \(M\) is divisible if \(aM=M\) for all \(a\in S\), while \(M\) is \(h\)-divisible if \(M\) is a homomorphic image of an injective \(R\)-module. For any \(R\)-module \(M, l(M) = \{x \in M | ax = 0 \text{ for some } a \in S\}\). A ring \(R\) is quasi-Frobenius if \(R\) is (left) Artinian and \(R\) is an injective \(R\)-module. A ring \(Q\) is a quotient ring of \(R\) if \(a^{-1}Q\) for all \(a \in S\) and given any \(q \in Q, q = a^{-1}b\) for some \(a \in S, b \in R\). Finally, the (left) singular ideal of \(R\) will be denoted by \(Z_f(R)\), while \(E(M)\) will denote the injective envelope of the \(R\)-module \(M\).

2. Main Results.

The following proposition is well-known and its proof is similar to that for commutative rings, (see, e.g. [6]).

PROPOSITION 2.1. Let \(R\) be a ring with quotient ring \(Q\). Then:

(a) Every injective \(R\)-module is divisible.

(b) Every torsion-free divisible \(R\)-module \(M\) is a \(Q\)-module and \(M\) is \(Q\)-injective if and only if \(M\) is \(R\)-injective.

We now extend Theorem 2.10 of [4] to non-commutative rings having a quasi-
Frobenius ring of quotients.

THEOREM 2.2. Let $R$ be a ring having a quasi-Frobenius quotient ring $Q$. If $M$ is an $h$-divisible $R$-module and $M/t(M)$ is an injective $R$-module, then $t(M)$ is a direct summand of $M$.

PROOF. Since $M/t(M)$ is a torsion-free injective $R$-module, $M/t(M)$ is an injective $Q$-module by Proposition 2.1. As $Q$ is quasi-Frobenius, every injective $Q$-module is a projective $Q$-module [1, p. 402]. Now $M$ is $h$-divisible hence, as in [4], there is an $R$-module $F$ which is isomorphic to a direct sum of copies of $Q$ and an epimorphism $\alpha : F \to M$. Thus $F$ is also a $Q$-module. Let $\beta : M \to M/t(M)$ be the natural homomorphism and let $f = \beta \alpha : F \to M/t(M)$. For any $q = a^{-1}b \in Q$, $x \in F$ we have

\[ a(\alpha(qx) + t(M)) = \alpha(bx) + t(M) = b(\alpha(x) + t(M)) \]

and so $f(qx) = qf(x)$; i.e., $f$ is a $Q$-homomorphism. Thus $K = \text{Ker} f$ is a $Q$-direct summand of $F$, say $F = K \oplus D$, and this is also a splitting of $F$ as an $R$-module. Now let $A = \alpha(D)$. We claim that $M = A \oplus t(M)$. For if $x \in A \cap t(M), x = \alpha(d)$ then $0 = \beta \alpha(d) = f(d)$ so $d = 0$ and hence $x = 0$. Also if $m \in M, m = \alpha(x)$ with $x = d + k$ where $d \in D, k \in K$ and so $m = \alpha(d) + \alpha(k)$. Since $0 = f(k) = \beta \alpha(k), \alpha(k) \in t(M)$ and so $M = A \oplus t(M)$. This completes the proof.

As a consequence we have the

COROLLARY. Let $R$ be a ring having a semisimple Artinian quotient ring $Q$. Then $t(M)$ is a direct summand of $M$ for every $h$-divisible $R$-module $M$.

PROOF. Since $M/t(M)$ is torsion-free and $h$-divisible (hence divisible) it is a $Q$-module and thus $Q$-injective. But then $M/t(M)$ is $R$-injective and the theorem applied.

We next consider a condition which ensures that a ring $R$ having an Artinian quotient ring will be semiprime. This result extends Theorem 3.3 of [4] to non-commutative rings.

THEOREM 2.3. Let $R$ be a ring having an Artinian quotient ring $Q$. The following conditions are then equivalent:

(a) $R$ is semiprime.

(b) $Q$ is semisimple.

(c) $E(M)/M$ is a torsion $R$-module for each $R$-module $M$.

PROOF. The equivalence of (a) and (b) follows from Small's Theorem [7] since $P(R) = R \cap P(Q)$ where $P(Q) - (P(R))$ denotes the prime radical of $Q(R)$.
If (b) holds then by [3, Thm. 3.9] every essential left ideal of \( R \) contains a non-zero divisor. Since \( M \) is essential in \( E(M) \), for any \( x \in E(M) \), \( (M : x) = \{ r \in R | rx \in M \} \) is an essential left ideal of \( R \). Thus there is a non-zero divisor \( a \in R \) for which \( ax \in M \) and so \( E(M)/M \) is a torsion \( R \)-module. Now assume (c) holds. First note that \( t(E(R)) = 0 \) since \( R \) is essential in \( E(R) \) and \( t(R) = 0 \). Thus \( \text{Hom}_R(E(R)/R, E(R)) = 0 \) and so the exact sequence \( 0 \to R \to E(R) \to E(R)/R \to 0 \) gives rise to the exact sequence \( 0 \to \text{Hom}_R(E(R), E(R)) \to \text{Hom}_R(R, E(R)) \to \text{Ext}^1_R(R, E(R)) = 0 \), and \( \text{Hom}_R(R, E(R)) \cong E(R) \). Thus \( E(R) \) has a ring structure compatible with the \( R \)-module structure. Moreover we can consider \( Q \subset E(R) = K \) so \( R \) has \( K \) as a ring of quotients and it can be verified that \( K \) is self-injective. Since \( R \) is finite-dimensional as an \( R \)-module, \( K \) is finite-dimensional as a \( K \)-module. Now if \( I \) is an essential left ideal of \( R \) then \( E(I) = E(R) \) so \( R/I \subset K/I \) hence \( R/I \) is torsion. Thus every essential left ideal of \( R \) contains a non-zerodivisor and so \( Z_I(R) = 0 \). But then \( Z_I(K) = 0 \) and so by [2, Thm. 1, p. 44], \( K \) is a regular finite-dimensional ring hence semisimple Artinian. Since \( P(R) = P(K) \cap R \), \( R \) is then semiprime. This completes the proof.

REMARK. We have shown above that \( E(R) \) is the maximal left quotient ring of \( R \) so that an alternate proof could be given by appealing to results of R. E. Johnson [5].

COROLLARY. Let \( R \) be a ring with a semisimple Artinian ring of quotients. If \( M \) is a torsion \( R \)-module then \( E(M) \) is a torsion \( R \)-module.

We conclude by giving a condition that ensures that a ring having an Artinian ring of quotients will have a quasi-Frobenius ring of quotients.

PROPOSITION 2.4. Let \( R \) be a ring with an Artinian quotient ring \( Q \). Then \( Q \) is quasi-Frobenius if and only if \( Q \) is an \( h \)-divisible \( R \)-module.

PROOF. If \( Q \) is quasi-Frobenius then \( Q \) is an injective \( Q \) or \( R \)-module. Thus suppose \( Q \) is \( h \)-divisible and let \( A \) be an injective \( R \)-module mapping onto \( Q \). Then for some index set \( I \), \( \bigoplus_{i \in I} R_i (R_i = R \text{ for all } i \in I) \), maps onto \( A \) and this mapping can be extended to an \( R \)-homomorphism from \( \bigoplus_{i \in I} E(R_i) = F \) onto \( Q \). As in theorem 2.2 \( E(R) \) is a \( Q \)-module and this last mapping is a \( Q \)-homomorphism. Since \( Q \) is \( Q \)-projective \( F = Q_0 \oplus P \) as \( Q \)-modules, with \( Q \approx Q_y \). Then \( Q_0 \), being cyclic, lies in \( F_0 = \bigoplus_{j=1}^k E(R_{i_j}) \) for some finite subset \( \{ i_1, \ldots, i_k \} \subset I \). Hence it follows that
$F_0 = Q_0 \oplus (P \cap F_0)$ and so $Q_0$ is $R$-injective and so $Q$ is $Q$- and $R$-injective by Proposition 2.1.

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