A NOTE ON THE GEOMETRIC MEANS OF ENTIRE FUNCTIONS
OF TWO COMPLEX VARIABLES

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1. Let

\[ f(z_1, z_2) = \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2} z_1^{k_1} z_2^{k_2}, \]

be an entire function of two complex variables \( z_1 \) and \( z_2 \), holomorphic for \( |z_t| \leq r_t, \ t=1, 2 \). We know that the maximum modulus of \( f(z_1, z_2) \) for \( |z_t| \leq r_t \) \((t=1, 2)\) is denoted as

\[ M(r_1, r_2) = \max_{|z_t| \leq r_t} |f(z_1, z_2)|, \ t=1, 2. \]

The finite order \( \rho \) of an entire function \( f(z_1, z_2) \) is denoted as ([1], p.219)

\[ \lim_{r_1, r_2 \to \infty} \sup \frac{\log \log M(r_1, r_2)}{\log(r_1 r_2)} = \rho. \]

The geometric means \( G(r_1, r_2) \) and \( g_k(r_1, r_2) \) of the function \( |f(z_1, z_2)| \) for \( |z_t| \leq r_t \) \((t=1, 2)\) have been defined as ([2])

\[ G(r_1, r_2) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| d\theta_1 d\theta_2 \right\} \]

and

\[ g_k(r_1, r_2) = \exp \left\{ \frac{(k+1)^2}{(r_1 r_2)^{k+1}} \int_0^{r_1} \int_0^{r_2} (x_1 x_2)^{k} \log G(x_1, x_2) dx_1 dx_2 \right\}, \]

where \( 0 < k < \infty \).

In this note I have investigated a few properties of the above defined geometric means.

2. Let \( \varphi(r_1, r_2) \) be a "slowly changing" function; that is \( \varphi(r_1, r_2) > 0 \) and continuous for \( r_1 > r_1^0, \ r_2 > r_2^0 \) and for every constants \( m, n > 0 \), \( \varphi(mx_1, nx_2) \sim \varphi(x_1, x_2) \) as \( x_1 \) or \( x_2 \) or \( x_1 \) and \( x_2 \) tend to infinity.

Also let us set

\[ \lim_{r_1, r_2 \to \infty} \sup \frac{\log G(r_1, r_2)}{(r_1 r_2)^{\theta} \varphi(r_1, r_2)} = c \quad (0 < d \leq c < \infty) \]
and

\[
\lim_{r_1, r_2 \to \infty} \sup_{r_1, r_2} \frac{\log g_k(r_1, r_2)}{q} = p \quad (0 < q \leq p < \infty).
\]

In my earlier paper ([2]), I have proved the following result:

If \( f(z_1, z_2) \) be an entire function of finite nonzero order \( \rho \), then

\[
\left\{ \frac{k+1}{k+\rho+1} \right\}^2 d \leq q \leq \left\{ \frac{k+1}{k+\rho+1} \right\}^2 c.
\]

Now I intend to prove the following theorems:

**THEOREM 1.** Let \( f(z_1, z_2) \) be an entire function of order \( \rho \), then

\[
\frac{k+1}{k+\rho+1} \leq \lim_{r_1, r_2 \to \infty} \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} \leq \frac{k+1}{k+\rho+1}
\]

**PROOF.** From (2.1) and (2.2), we obtain

\[
\frac{q-\varepsilon}{c+\varepsilon} \leq \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} \leq \frac{p+\varepsilon}{d-\varepsilon}.
\]

Taking limits and using (2.3), the result follows.

**COROLLARY.** If \( c=d \), then

\[
(k+1)^2 \log G(r_1, r_2) \sim (k+\rho+1)^2 \log g_k(r_1, r_2).
\]

**THEOREM 2.** Let \( f(z_1, z_2) \) be an entire function and if \( 0 < r_1 < R_1, \ 0 < r_2 < R_2 \), then

\[
[(R_1 R_2)^{k+1} - (r_1 r_2)^{k+1}] \log G(r_1, r_2) \leq [(R_1 R_2)^{k+1} \log g_k(R_1, R_2)]
\]

\[
- (r_1 r_2)^{k+1} \log g_k(r_1, r_2) \leq (R_1 R_2)^{k+1} - (r_1 r_2)^{k+1} \log G(R_1, R_2)
\]

**PROOF.** Since \( G(r_1, r_2) \) is an increasing function of \( r_1 \) and \( r_2 \), therefore from (1.2) we have

\[
(R_1 R_2)^{k+1} \log g_k(R_1, R_2) - (r_1 r_2)^{k+1} \log g_k(r_1, r_2)
\]

\[
= (k+1)^2 \int \int (x_1 x_2)^k \log G(x_1, x_2) dx_1 dx_2
\]

\[
\leq [(R_1 R_2)^{k+1} - (r_1 r_2)^{k+1}] \log G(R_1, R_2).
\]

Also

\[
(R_1 R_2)^{k+1} \log g_k(R_1, R_2) - (r_1 r_2)^{k+1} \log g_k(r_1, r_2)
\]

\[
= (k+1)^2 \int \int (x_1 x_2)^k \log G(x_1, x_2) dx_1 dx_2
\]
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\[ \geq [(R_1 R_2)^{k+1} - (r_1 r_2)^{k+1}] \log G(r_1, r_2). \]

Hence the result follows.

**COROLLARY.** If \( \eta \) (0 < \( \eta \) < 1) is a constant, then

\[ \lim_{r_1, r_2 \to \infty} \left[ \frac{\{g_k(\beta r_1, \beta r_2)\}^{\frac{1}{(k+1)^{b}}}}{g_k(r_1, r_2)} \right] = 0. \]

Putting \( r_1 = \beta r_1 \), \( r_2 = \beta r_2 \) and \( R_1 = r_1 \), \( R_2 = r_2 \) in (2.4) and taking the limit the result follows.

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**REFERENCES**
