ON THE NON-RIEMANNIAN SPACES

By OM. P. Singh

§ 1. Introduction.

In an $n$-dimensional non-Riemannian space $A_n$ the theory of normal tensors based on the system of normal coordinates introduced by O. Veblen [1], have been developed by O. Veblen and T.Y. Thomas [2]. In the present paper, considering the space $A_n$ embodied with a field of normal tensors $A_{jk}^i$, we shall establish some theorems on the space and shall deal with the affine motion in an $A_n$. We shall also determine the necessary and sufficient conditions for the equations of the paths admitting homogeneous LFI and QFI, and for the existence of the infinitesimal projective collineation in the space $A_n$.

The symbolism employed in the present paper is essentially the same as these used in [1], [2] and by Eisenhart [4].

§ 2. The field of normal tensors $A_{jk}^i$ and the space $A_n$.

Considering the symmetric functions $I_{jk}^i$ and $C_{jk}^i$ in the space $A_n$ as the connection coefficients in the coordinates $x$'s and its normal coordinates $y$'s respectively, the normal tensors $A_{jk}^i, \ldots$ are given by

$$A_{jk, \ldots}^i = \left( \frac{\partial C_{jk}^i}{\partial y^j \cdots \partial y^i} \right)_0,$$

where the lower suffix zero indicates the value of the function evaluated at the point $P_0$ as origin of the normal coordinates $y$'s. From (2.1), it is quite obvious that the components of the normal tensors $A_{jk}^i, \ldots$ are symmetric in $j$ and $k$, and in last $r$ indices. Veblen and Thomas have already shown that any point in the space can be chosen as origin, so the components of these tensors are defined at each point throughout the space and in every coordinate system.

Now for the sake of convenience in our very purpose of present discussion, 

1). Numbers in brackets refer to the references at the end of the paper.
2). Eisenhart [4], pp. 68—74.
we take the 4th order mixed normal tensors $A_{jkl}^i$ in consideration and assume that
each point of the space $A_n$ is defined by the tensors $A_{jkl}^i$. So that the space $A_n$
is now embodied with a field of normal tensors $A_{jkl}^i$. This idea has motivated the
author to discuss the various problems in this characteristic space.

The components of the normal tensors $A_{jkl}^i$ in terms of the connection coefficient $\Gamma_{jk}^i$ are defined by the expression

\begin{equation}
\tag{2.2}
A_{jkl}^i = \frac{\partial \Gamma_{jk}^i}{\partial x^l} - \Gamma_{jk}^i \Gamma_{jl}^\beta - \Gamma_{j}^\beta \Gamma_{kl}^i,
\end{equation}

where

\begin{equation}
\tag{2.3}
\Gamma_{jk}^i = \frac{1}{3} P \left( \frac{\partial \Gamma_{jk}^i}{\partial x^l} - \Gamma_{rk}^i \Gamma_{jl}^r - \Gamma_{j}^r \Gamma_{kl}^r \right),
\end{equation}

\begin{equation}
= \frac{1}{3} P \left( \frac{\partial \Gamma_{jk}^i}{\partial x^l} - 2 \Gamma_{jr}^i \Gamma_{kl}^r \right).
\end{equation}

In above relation $P$ before an expression indicates the sum of terms obtained
by permuting the subscripts cyclically.

We also notice that the tensors $A_{jkl}^i$ satisfy the following identity ([4], p. 70)

\begin{equation}
\tag{2.4}
A_{jkl}^i + A_{kjl}^i + A_{jlk}^i = 0.
\end{equation}

The Curvature tensor $B_{jkl}^i$ of the space $A_n$ is given by

\begin{equation}
\tag{2.5}
B_{jkl}^i = \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l} + \Gamma_{kl}^h \Gamma_{hi}^i - \Gamma_{jh}^i \Gamma_{kl}^h.
\end{equation}

Since $\Gamma_{jk}^i$, as defined by (2.3) is symmetric in $k$ and $l$, so it follows from
(2.2) and (2.5) that

\begin{equation}
\tag{2.6}
B_{jkl}^i = A_{jlk}^i - A_{jkl}^i.
\end{equation}

Now in accordance with the definition of flat spaces, if the space $A_n$ be flat one, i.e. $B_{jkl}^i = 0$, then we have $A_{jlk}^i = A_{jkl}^i$. Applying this fact to the identity
(2.4), we get $A_{jkl}^i = 0$. Hence we have the

**THEOREM 1.** If a non-Riemannian space $A_n$ be a flat space, i.e. $B_{jkl}^i = 0$, there
does not exist a field of normal tensors $A_{jkl}^i$ in an $A_n$, and conversely if the space
A_n embodies a field of normal tensors \( A_{jkl}^i \), the space \( A_n \) is necessarily a non-flat space.

Further we remark at this point that a flat space is necessarily a Riemannian space ([4], p. 81), which asserts that the normal coördinates of a flat \( A_n \) are now the particular class of Riemannian coördinates in a general Riemannian space. Hence it follows:

**THEOREM 2.** There does not exist a field of normal tensors \( A_{jkl}^i \) in a general Riemannian space \( V_n \).

§3. A non-Riemannian space of recurrent curvature and the affine motion.

Due to Y.C. Wong [3], an \( n \)-dimensional analytic non-Riemannian space \( A_n \) of recurrent curvature, for a symmetric affine connection \( \Gamma^i_{jk} \), has been defined by the characteristic relation

\[
B^i_{jkl;m} = K_m B^i_{jkl}, \quad (K_m \neq 0),
\]

where a semi-colon followed by an index denotes the covariant differentiation with respect to \( x^i \)'s. Such a space \( A_n \) is called an \( AK_n^* \)-space.

On the other hand, let us consider a field of normal tensors \( A_{jkl}^i \), characterized by a similar type of relation

\[
A^i_{jkl;m} = K_m A_{jkl}^i,
\]

where \( K_m \) is a non-zero covariant vector. We shall call such a field, a field of recurrent normal tensors.

Differentiating the relation (2.6) covariantly with respect to \( x^m \), we obtain

\[
B^i_{jkl;m} = A^i_{jkl;m} - A_{jkl;m}^i.
\]

If we introduce the relation (3.2) in (3.3), then, on applying the relation (2.6) in the so obtained result, we can get at once the relation (3.1). Consequently, we have the

**THEOREM 3.** When a non-Riemannian space \( A_n \) embodies a field of normal tensors \( A_{jkl}^i \), the space \( A_n \) is necessarily a space of recurrent curvature if the field possesses the recurrence property.

Further, we have an useful relation ([4], p. 72)

\[
A^i_{jkl;m} = A^i_{jkl;m} - A_{(jkl)m}^i.
\]
where
\[ A^i_{(jkl)m} = \left( \partial C^i_{jklm} / \partial x^m \right)_0. \]
In consequence of this, relation (3.3) is changed into the form
\[ B^i_{jkl;m} = A^i_{jkm} - A^i_{jklm'} \quad (\text{Because } C^i_{jkl} = C^i_{jkl}). \]

We now remember that a space \( A_n \) with the property that the curvature tensor of \( A_n \) are covariant constant, i.e., \( B^i_{jkl;m} = 0 \), is said to be symmetric in the sense of Cartan. In such a case, from above relation, we have the result \( A^i_{jkm} = A^i_{jklm} \), which shows that the tensor \( A^i_{jkm} \) is symmetric in the middle two indices \( l \) and \( k \). But, it has already been pointed out that a normal tensor \( A^i_{jkl,...} \), is symmetric in first two indices \( j \) and \( k \) and in last \( r \) indices, so, in the case when \( A_n \) is symmetric, the tensor \( A^i_{jklm} \) is throughout symmetric in its all lower indices. However, we have the following well known identity ([4], p.70) satisfied by the tensors \( A^i_{jklm} \)

\[ A^i_{jklm} + A^i_{jmk} + A^i_{jml} + A^i_{kml} + A^i_{kim} + A^i_{lmj} = 0. \]

In our present case, from the above facts and this identity, we conclude that \( A^i_{jklm} = 0 \). Hence, we can state the

**THEOREM 4.** When a non-Riemannian space \( A_n \) is a symmetric space, there does not exist a field of normal tensors \( A^i_{jklm} \) in an \( A_n \), and conversely, if there does not exist a field of normal tensors \( A^i_{jklm} \) in an \( A_n \), the space \( A_n \) is necessarily a symmetric space.

**COROLLARY 1.** If an \( A_n \) embodies a field of normal tensors \( A^i_{jklm} \) the space \( A_n \) be a non-symmetric space.

Next, in the space \( A_n \), let us consider an infinitesimal transformation.

\[ (3.4) \quad x^i = x^i + \xi^i(x) \delta \tau, \]

where \( \xi^i \) is a contravariant vector, and \( \delta \tau \) is an infinitesimal constant.

Then we have a deformed space with an affine connection

\[ T^i_{jk} \overset{\text{def}}{=} T^i_{jk} = T^i_{jk} + (\xi I^i_{jk}) \delta \tau, \]

where \( T^i_{jk} \) is the affine connection of the deformed space.

In case of the above transformation, if the original space and the deformed space have the same connection, the transformation (3.4) is called an affine motion of the space \( A_n \). In order that it be the case, it is necessary and sufficient
that we have

\( \mathcal{L} \Gamma^i_{jk} = 0 \) \(^3\),

where the notation \( \mathcal{L} \) denotes the operator of the so called Lie differentiation process.

The Lie derivative of the curvature tensor \( B^i_{jkl} \) is given by \( \mathcal{L} B^i_{jkl} = (\mathcal{L} \Gamma^i_{jl}) : k - (\mathcal{L} \Gamma^i_{jk}) : l \). So, due to the condition (3.5), under an affine motion we get necessarily

\( \mathcal{L} B^i_{jkl} = 0. \)

Now, operating \( \mathcal{L} \) on both the sides of (2.6) and applying the condition (3.6) in the result, we obtain

\( \mathcal{L} A^i_{jkl} = \mathcal{L} A^i_{jkl}. \)

Thus, it concludes the

THEOREM 5. When a non-Riemannian space \( A_n \) embodies a field of normal tensors \( A^i_{jkl} \), in order that the transformation (3.4) be an affine motion of \( A_n \), it is necessary and sufficient that we have

\( \mathcal{L} A^i_{jkl} = \mathcal{L} A^i_{jkl}. \)

Further we see that for any general tensor \( T^i_{jkh} \), the following relation \(^4\) holds good

\( \mathcal{L} T^i_{jkh} = \mathcal{L} T^i_{jkh} \)

In the present case, employing this formula for the curvature tensor \( B^i_{jkl} \), we get from (3.5) and (3.6)

\( \mathcal{L} (B^i_{jkl} : m) = 0. \)

Introducing the relation (3.3) in (3.9) and using (3.2) and (3.7), we obtain after some simple calculation

\( (A^i_{l,k} - A^i_{jkl}) \mathcal{L} K_m = 0. \)

Since our space is not flat, i.e. \( B^i_{jkl} - A^i_{jkl} \neq 0 \), therefore \( \mathcal{L} K_m = 0. \)

Now, if we again use the relation (3.8) for the tensor \( A^i_{jkl} \) and apply the relat-

\(^3\) See Yano [5], relation (2.21), page 8.

\(^4\) Yano [5], page 16.
ion (3.2) and the condition $\xi K_m = 0$ in the result, then, under an affine motion, we get a requisite condition $(\xi A_{ijkl}^i)_m = K_m(\xi A_{ijkl}^i)$. Hence, we have the

**THEOREM 6.** When an $AK_n$-space embodies a field of recurrent normal tensors $A_{ijkl}^i$, in order that the field may admit an affine motion of the form (3.4), it is necessary and sufficient that $(\xi A_{ijkl}^i)_m = K_m(\xi A_{ijkl}^i)$.

§ 4. The equations of the paths admitting independent homogeneous linear first integrals and quadratic first integrals.

In an $n$-dimensional space $A_n$, let us consider the equations of the paths

$$(4.1) \quad \frac{d^2 x^i}{ds^2} + f_{jk}^{il} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$ 

If each integral of the equations of the above paths satisfies the condition

$$a_{ijkl} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} \cdots \frac{dx^n}{ds} = C; \quad (C: \text{a constant}),$$

the equations (4.1) are said to admit a homogeneous first integral of the $m$th degree. But for the sake of brevity, here on one hand we shall consider the integral of the first degree, that is an independent homogeneous linear first integral (LFI)

$$(4.2) \quad a_i \frac{dx^i}{ds} = C_1; \quad (C_1: \text{a constant}).$$

If the equations of the paths (4.1) admit a homogeneous LFI (4.2), it is necessary that

$$(4.3) \quad a_{ii} + a_{ij} = 0.$$ 

Differentiating this relation covariantly with respect to $x^j$, we have

$$(4.4) \quad a_{ij} + a_{ij} = 0.$$ 

Now, we consider that $T_{ijkl}^i$ and $T_{ijkl}^i$ are the components of a tensor in the coordinate system $x^i$ and its normal coordinates $y^i$ respectively, then the general relation ([4], p. 74) connecting covariant derivatives of the tensor $T_{ijkl}^i$, and its extensions of the second order can be given as follows:

$$(4.5) \quad T_{ijkl}^i = T_{ijkl}^i + \sum_{p=1}^r T_{ijkl}^i \frac{x^p}{x^i} A_{ijkl}^{ip} - \sum_{q=1}^s T_{ijkl}^i \frac{x^q}{x^i} A_{ijkl}^{jq}.$$ 

where $T_{ijkl}^i$ are the second order extensions of the tensor $T_{ijkl}^i$, defined by
(4.6) \[ T_{i_1 \ldots i_n}^{j_1 \ldots j_m} = \left( \frac{\partial^2 T_{i_1 \ldots i_n}^{j_1 \ldots j_m}}{\partial y_{j_1} \partial y_{j_m}} \right)_0. \]

The colon followed by the indices denotes the extension process of the tensors of the same order as the sum of the given number of indices following a colon, and the lower suffix zero indicates the value of the function evaluated at the point \( P_0 \) as origin. Since any point can be chosen as \( P_0 \), so the relation (4.5) holds good for any point throughout the space.

Similarly if we assume that \( a_i \) and \( \alpha_i \) are the components of a tensor of order one in \( x^i \) coordinates and its normal coordinates \( y^j \) respectively, then, with the help of (4.5) and (4.6) the covariant derivatives of \( a_i \) in terms of the second order extension can be given by

(4.7) \[ a_{i;jk} = a_{i;jk} - a_k A_{i;jk}^h, \]

where

\[ a_{i;jk} = \left( \frac{\partial^2 a_i^h}{\partial y^j \partial y^k} \right)_0. \]

Now, adding the relation (4.4) with the first of the following,

\[ a_{k;ij} + a_{i;kj} = 0, \quad a_{j;ki} + a_{k;ji} = 0 \]

and subtracting the other, then, on making the use of the relation (4.7) and the identity (2.4), we shall get at once

(4.8) \[ a_{i;jk} = -a_k (A_{j;ki} + A_{k;ji}) \]

Making the cyclic change in the indices \( i, j, k, \) of the relation (4.8) and adding the so obtained two relations with that of (4.8), we get at last

(4.9) \[ a_{i;jk} + a_{j;ki} + a_{k;ij} = 0, \]

where we have used the identity (2.4) again. Hence we have the

**THEOREM 7.** When a non-Riemannian space \( A_n \) embodies a field of normal tensors \( A_{j;ki}^h \), in order that the equations of the paths (4.1) of the space \( A_n \) admit homogeneous LFI (4.2), it is necessary and sufficient that the identity (4.9) necessarily holds good.

Next, we consider the integral of the second degree, that is an independent homogeneous quadratic first integral (QFI)

(4.10) \[ a_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = C_2; \quad (C_2: \text{a constant}). \]

If the equations of the paths (4.1) admit a homogeneous QFI (4.10), it is necessary that
Now, for the sufficiency that the equations of the paths (4.1) admit a homogeneous QFI (4.10), we determine a concrete condition. For this purpose, first, differentiating (4.11) covariantly, we obtain

\[(4.12) \quad a_{ij;kl} + a_{jk;il} + a_{ki;jl} = 0.\]

Next, by making the use of the formula (4.5), one may easily have the

\[(4.13) \quad a_{ij;kl} = a_{ij;kl} - a_{kj} A_{jl}^h - a_{il} A_{jk}^h.\]

Similarly, for the other terms in (4.12), consecutive expressions are obtained as

\[(4.14) \quad a_{jk;il} = a_{jk;il} - a_{kl} A_{ji}^h - a_{lj} A_{ki}^h,\]

\[(4.15) \quad a_{ki;jl} = a_{ki;jl} - a_{li} A_{kj}^h - a_{jl} A_{ki}^h.\]

Here we notice that the second extension \(a_{ij;kl}\) of the tensor \(a_{ij}\) is symmetric in \(i\) and \(j\), and in \(k\) and \(l\). Also the normal tensor \(A_{jk}^h\) is symmetric in its first two lower indices. So adding (4.13), (4.14), (4.15) and using (4.12), we can get with ease

\[(4.16) \quad a_{ij;kl} + a_{jk;il} + a_{ki;jl} = 2(a_{kj} A_{jl}^h + a_{kl} A_{ji}^h + a_{ki} A_{lj}^h).\]

Making the cyclic change of the indices \(i, j, k\) and \(l\) in (4.16), three other expressions will be obtained successively. Adding these three with that of (4.16) and using the identity (2.4), and on arranging the terms, we get finally

\[(4.17) \quad (a_{ij;kl} + a_{kl;ij}) + (a_{jk;il} + a_{jl;ki}) + (a_{ki;jl} + a_{lj;ki}) = 0.\]

From which, obviously we see that, if \(a_{ij;kl} = -a_{kl;ij}\), the relation (4.17) is identically satisfied. Hence we have the

**THEOREM 8.** When a non-Riemannian space \(A_n\) embodies a field of normal tensors \(A_{jk}^i\), in order that the equations of the paths (4.1) of the space \(A_n\) admit homogeneous QFI (4.10), it is necessary and sufficient that \(a_{ij;kl} = -a_{kl;ij}\), that is to say, the second extension \(a_{ij;kl}\) of the tensor \(a_{ij}\) be antisymmetric in its pair of indices \(ij\) and \(kl\).

§5. Conditions for the existence of infinitesimal collineations in an affinely connected space.

If some point transformations transform the points of an affinely connected manifold into points of the manifold such that the paths are transformed into paths. The transformations are called collineations.
Let us consider an infinitesimal point transformation

$$\mathbf{x}' = \mathbf{x} + \xi(x) \partial \tau,$$

where $\xi$ are the function of $x$'s and $\partial \tau$ is an infinitesimal. If (5.1) be the collineation in an $A_n$, the necessary conditions for this are

$$\xi^h = \xi^h B_{ij}^h + \partial^h_i \varphi_j + \partial^h_j \varphi_i,$$

and

$$\xi^h = \xi^h D_{ij}^h.$$

When a set of functions $\xi^i$ are a solution of (5.2), where $\varphi_j \neq 0$, the collineation preserves the projective properties and is called an infinitesimal projective collineation. On the other hand, when the set of functions $\xi^i$ are a solution of (5.3) and the collineation (5.1) preserves the affine properties of the space, the collineation (5.1) is called an infinitesimal affine collineation in the space $A_n$.

The second order covariant derivative of a vector $\xi^i$ and its second extension may be connected by the relation

$$\xi_{ij}^h = \xi_{ij}^h + \xi^h A_{ij}^h.$$

Introducing this relation on the left of (5.2) and taking help of (2.6), the relation (5.2) may be rewritten in the form

$$\xi_{ij} + \xi_{ij} A_{ij}^h = \xi^h (A_{ij}^h - A_{ij}^h) + \partial^h_i \varphi_j + \partial^h_j \varphi_i.$$

Interchanging the indices $i$ and $j$ in this relations, and adding the resulting equation with (5.6), we get

$$\xi_{ij} + \xi_{ij} A_{ij}^h = \xi^h (A_{ij}^h - A_{ij}^h + A_{ki}^h - A_{ki}^h) + 2(\partial^h_i \varphi_j + \partial^h_j \varphi_i).$$

or

$$\xi_{ij} + \xi_{ij} = \xi^h (A_{ij}^h - A_{ij}^h + A_{ki}^h - A_{ki}^h) + 2(\partial^h_i \varphi_j + \partial^h_j \varphi_i).$$

Since the tensors $A_{ij}^h$ are symmetric in the first two lower indices $i$ and $j$, so because of the condition $\xi_{ij} = \xi_{ji}$ from (5.7) we shall obtain

$$\xi_{ij} = -\xi^h A_{ij}^h + \partial^h_i \varphi_j + \varphi_i \varphi_j.$$

Contracting the relation (5.8) with respect to the indices $h$ and $i$, we find that

$$0 = -\xi^h A_{jik}^h + (n+1)\varphi_j.$$

From which we conclude that

$$\varphi_j = \frac{1}{n+1} \xi^h A_{jik}^h. \quad \text{(Because } A_{jik}^h = A_{jki}^h).$$
Hence, we can summarize the above result as follows:

**Theorem 9.** A non-Riemannian space \( A_n \) embodied with a field of normal tensors \( A^h_{ijk} \) admits an infinitesimal projective collineation if and only if the covariant vector \( \varphi_j \) satisfies the relation

\[
\varphi_j = \frac{1}{n+1} \xi^k A_{jkh}^h.
\]

Next, we remark that, when \( \varphi_j \) is zero, the projective collineation becomes an affine collineation. In such a case, we conclude that \( \xi^k A_{jkh}^h = 0 \). Consequently, it follows

**Corollary 2.** When a non-Riemannian space \( A_n \) embodies a field of normal tensors \( A^h_{ijk} \) in order that an infinitesimal projective collineation admitted in \( A_n \) becomes an infinitesimal affine collineation in the space, it is necessary and sufficient that we assume \( \xi^k A_{jkh}^h = 0 \).

Furthermore, we shall determine a concrete result for the existence of the infinitesimal collineation in the space \( A_n \). For this very purpose, first we have to assume \( \xi^k = \eta_i^h \).

Introducing this concept in (5.2) and using the relations (2.6) and (5.9), we find that

\[
\eta_{lij}^h = -\xi^l (A^h_{lij} - A^h_{i lj}) + \frac{1}{n+1} [\delta^h_j \xi^l A^r_{irl} + \delta^h_i \xi^l A^r_{jrl}].
\]

Differentiating this relation covariantly with respect to \( x^k \), then with the help of relation \( \xi^k = \eta_i^h \), we obtain

\[
(5.10) \quad \eta_{ijkl}^h = \eta_{i}^{l} (A_{lij}^h - A_{ijl}^h) + \xi^{l} (A_{ijl; k}^h - A_{ijl; k}^h) + \frac{1}{n+1} [\delta^h_j (\eta_{k}^{l} A^r_{irl} + \xi^l A^r_{irl; k})
\]

\[ + \delta^h_i (\eta_{k}^{l} A^r_{jrl} + \xi^l A^r_{jrl; k})].
\]

Also, we have the following relation connecting the second order covariant derivative \( \eta_{i j k}^h \) of a tensor \( \eta_i^h \) and its second extension \( \eta_{i j k}^h \),

\[
(5.11) \quad \eta_{i j k}^h = \eta_{i j k}^h + \eta_{i j k}^h A_{i j k}^h - \eta_{i j k}^h A_{i j k}^h.
\]

Introducing the relation (5.11) on the left of (5.10), we have

\[
\eta_{i j k}^h + \eta_{i j k}^h A_{i j k}^h - \eta_{i j k}^h A_{i j k}^h = \eta_{i}^{l} (A_{lij}^h - A_{ijl}^h) + \xi^{l} (A_{ijl; k}^h - A_{ijl; k}^h)
\]

\[ + \frac{1}{n+1} [\delta^h_j (\eta_{k}^{l} A^r_{irl} + \xi^l A^r_{irl; k}) + \delta^h_i (\eta_{k}^{l} A^r_{jrl} + \xi^l A^r_{jrl; k})].
\]
Now, if we interchange the indices $j$ and $k$ in this relation, then on subtracting the later equation with the former one, we get at last

$$
\eta^i_{ik} A^h_{ijk} - \eta^i_{ij} A^h_{ijk} - \eta^i_{ik} A^h_{ijk} + \eta^i_{ik} A^h_{ijk} = \eta^i_{ik} (A^h_{ijk} - A^h_{ijk})
$$

where we have used $\eta^i_{i;k} = \eta^i_{i;k}$. The contraction of $h=i$ in this equation gives us

$$
(5.13)
\begin{align*}
\eta^i_{r} A^r_{ijk} - \eta^i_{r} A^r_{ijk} - \eta^i_{r} A^r_{ijk} + \eta^i_{r} A^r_{ijk} &= \eta^i_{r} A^r_{ijk} + \xi^i_{r} A^r_{ijk} - \eta^i_{r} A^r_{ijk} - \xi^i_{r} A^r_{ijk;f} \\
\eta^i_{f} A^f_{ijk} + \xi^i_{f} A^f_{ijk;f} &= \eta^i_{f} A^f_{ijk} + \xi^i_{f} A^f_{ijk;f}
\end{align*}
$$

Hence, we have the

**Theorem 10.** When a non-Riemannian space $A_n$ embodies a field of normal tensors $A^h_{ijk}$, in order that the space $A_n$ admits an infinitesimal projective collineation, it is necessary and sufficient that we have

$$(\xi^i_{r} A^r_{ijk}) : k = (\xi^i_{r} A^r_{ijk}) : j$$

In concluding this paper, the author is very much indebted to Dr. H.D. Singh due to his kind help. The author also wishes to express his many thanks to Prof. K. Takano (Tokyo) for going through the manuscript of this paper and for many valuable suggestions.

R.B.S. College, Agra (India).

**References**


