INTEGRALS INVOLVING LAGUERRE, JACOBI AND HERMITE POLYNOMIALS

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Summary.
The purpose of the present paper is to evaluate certain integrals involving Laguerre, Jacobi and Hermite polynomials. These integrals are very useful in case of expansion of any polynomial in a series of Orthogonal polynomials [1, Theo.56]. Frequent use will be made of the notations given in Rainville [1].

We shall prove the following:

1. \[ \int_0^\infty e^{-x}x^{\alpha+\beta}L_K^{(\alpha)}(x)dx = \frac{(-1)^K\Gamma(1+\alpha+\beta)\Gamma(1+\beta)}{K!\Gamma(1+\beta-K)}, \]
   where Re(\alpha+\beta) > -1 and 0 ≤ K ≤ β:

2. \[ \int_{-1}^1 (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+Y}\Gamma(n+\alpha+1)\Gamma(\beta+Y+1)}{n!\Gamma(\alpha+\beta+Y+n+2)} \times \frac{\Gamma(Y+1)}{\Gamma(Y-n+1)}, \]
   where Re(\alpha) > -1 and Re(\beta+Y) > -1:

3. \[ \int_{-\infty}^{\infty} e^{-x^2}H_{2n}(x)dx = \frac{2^{2n}\Gamma(l+1)\Gamma(l+1/2)}{\Gamma(l-n+1)} \]
   where 0 ≤ n ≤ l:

4. \[ \int_{-\infty}^{\infty} e^{-x^2}H_{2n+1}(x)dx = \frac{2^{2n+1}\Gamma(l+1)\Gamma(l+3/2)}{\Gamma(l-n+1)}, \]
   where 0 ≤ n ≤ l.

PROOF of (1).
Consider the integral

\[ A = \int_0^\infty e^{-x}x^{\alpha+\beta}L_K^{(\alpha)}(x)dx. \]  (1.1)

From [1, p. 201(3)] and (1.1), we have

\[ A = \sum_{r=0}^{K} \frac{(-1)^r\Gamma(1+\alpha)_{\kappa}}{r!(K-r)!\Gamma(1+\alpha)_r} \int_0^\infty e^{-x}x^{\alpha+\beta+r}dx, \]
If \( \text{Re}(\alpha + \beta) > -1 \), then
\[
A = \sum_{r=0}^{K} \frac{(-1)^r(1+\alpha)_K}{r!(K-r)!(1+\alpha)_r} \Gamma(\alpha+\beta+r+1),
\]
or,
\[
A = \frac{(1+\alpha)_K \Gamma(\alpha+\beta+1)}{K!} \sum_{r=0}^{K} \frac{(-K)_r(\alpha+\beta+1)_r}{r!(1+\alpha)_r}.
\]
If \( 0 \leq K \leq \beta \), then
\[
A = \frac{(-1)^K \Gamma(\alpha+\beta+1) \Gamma(\beta+1)}{K! \Gamma(\beta+K+1)}.
\]
Hence the result follows.

**PROOF of (2).**
Consider the integral
\[
B = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta + Y P_n^{(\alpha, \beta)}(x) \, dx.
\]  
From [1, p. 255(4)] and (2.1), we have
\[
B = \sum_{k=0}^{n} \frac{(-1)^K (1+\alpha)_n (1+\alpha+\beta)_n + K}{K!(n-K)!(1+\alpha)_K (1+\alpha+\beta)_n} \Gamma(\alpha+\beta+Y+1) \int_{-1}^{1} (1-x)^\alpha+K (1+x)^\beta+Y \, dx.
\]
If \( \text{Re}(\alpha) > -1 \) and \( (\beta+Y) > -1 \), then
\[
B = 2^{\alpha+\beta+Y+1} \frac{\Gamma(1+\alpha+n) \Gamma(\beta+Y+1)}{n! \Gamma(\alpha+\beta+Y+2)} \sum_{k=0}^{n} \frac{(-1)^K (1+\alpha+\beta+n)_K}{K!(\alpha+\beta+Y+2)_K} \Gamma(\alpha+n+1) \Gamma(\beta+Y+1) (-1)^n (-Y)_n
\]
\[
= 2^{\alpha+\beta+Y+1} \frac{\Gamma(1+\alpha+n+1) \Gamma(\beta+Y+1) \Gamma(Y+1)}{n! \Gamma(\alpha+\beta+Y+n+2) \Gamma(Y-n+1)}
\]
Hence the required result.

**PROOF of (3).**
Consider the integral
\[
C = \int_{-\infty}^{\infty} e^{-x^2} x^{2l} H_{2n}(x) \, dx.
\]  
From [1, p. 187(2)] and (3.1), we have
\[
C = \sum_{k=0}^{n} \frac{(-1)^K (2n)! 2^{2n-2K}}{(2n-2K)!} \int_{-\infty}^{\infty} e^{-x^2} x^{2l+2n-2K} \, dx
\]
\[
= 2^{2n} \Gamma(l+n+1/2) \sum_{k=0}^{n} \frac{(-n)_K (-n+1/2)_K}{K!(1/2-l-n)_K}.
\]
If \( 0 \leq n \leq l \), then
\[
C = 2^{2n} \frac{\Gamma(l+1)\Gamma(l+1/2)}{\Gamma(l-n+1)}.
\]
Hence the precise result.

The proof of (4) is similar to that of (3).

Acknowledgement.
I wish to express my grateful thanks to Dr. (Mrs) Pramila Srivastava for her kind help and guidance during the preparation of this paper.

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REFERENCE