SHIROTA'S THEOREM FOR N-COMPACT SPACES

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We shall consider only Hausdorff spaces.

In [8] Shirota has proved the following theorem: if a completely regular space \( X \) admits a complete uniformity and every closed discrete subspace of \( X \) is realcompact then \( X \) is realcompact. In this paper, we shall give an elementary proof of an analogous result of Shirota's theorem concerning \( N \)-compact spaces.

\( N \)-compact spaces were introduced by S. Mrowka in [4], where the general concept of an \( E \)-compact space was defined: given a space \( E \), a space \( X \) is \( E \)-compact (respectively \( E \)-completely regular) if it is homeomorphic to a closed subspace (respectively, subspace) of \( E^m \) for some cardinal number \( m \). For every \( E \)-completely regular space \( X \) there exists a unique (up to homeomorphism) \( E \)-compactification \( \beta_EX \) of \( X \) such that every continuous function \( f \) of \( X \) into an \( E \)-compact space \( Y \) admits a continuous extension \( f^\ast \) from \( \beta_EX \) into \( Y \) ([6], theorem 4.14). Thus, \( I \)-compact spaces (where \( I \) is the closed unit interval) are compact Hausdorff spaces; the \( R \)-compact spaces are the realcompact spaces or \( Q \)-spaces and \( \beta_IX, \beta_RX \) are respectively the Stone-Čech compactification and the Hewitt-realcompactification of a completely regular space \( X \). For \( D = \{0, 1\} \) a two point discrete space; \( N \) the discrete space of all positive integers = the countably infinite discrete space; \( N^\ast \) the one-point compactification of \( N \). It is easy to derive from the definition that the following conditions are equivalent: a) \( X \) is 0-dimensional Hausdorff (“0-dimensional” means “having a base of clopen sets”); b) \( X \) is \( D \)-completely regular; c) \( X \) is \( N \)-completely regular; and d) \( X \) is \( N^\ast \)-completely regular. It is also clear that every \( N \)-compact space is 0-dimensional realcompact and recently, it was shown in [7] that not every 0-dimensional realcompact space is \( N \)-compact. But, the realcompactness or \( N \)-compactness of a discrete space depends only upon the cardinality of the space, namely [2], [5].

PROPOSITION. For a discrete space \( X \), the following are equivalent.

a) \( X \) is realcompact
b) \( X \) is \( N \)-compact
c) The cardinality of \( X \) is non-measurable

We recall that a system of subsets of a space is said to be discrete provided that
each point of the space has a neighborhood which intersects at most one member of the system. If \( \{ F_a : a \in A \} \) is a discrete system of subsets of a space \( X \) and \( p_a \in F_a \) for each \( a \in A \), then \( \{ p_a : a \in A \} \) is a closed discrete subspace of \( X \). Therefore, according to the above proposition Shirota's theorem can be formulated as follows:

If a completely regular space \( X \) admits a complete uniformity and has the property that

\((\ast)\) the cardinality of every discrete system of subsets of \( X \) is non-measurable, then \( X \) is realcompact.

The analogous result for Shirota's theorem concerning \( N \)-compact spaces reads as follows.

**THEOREM.** If an ultranormal space \( X \) admits a complete uniformity of clopen coverings and \( X \) has the property \((\ast)\), then \( X \) is \( N \)-compact.

(Recall that a space \( X \) is ultranormal if for any two disjoint closed sets \( F, G \) in \( X \) there exists a clopen subset \( U \) of \( X \) such that \( F \subset U \) and \( U \cap G = \emptyset \). The above theorem, in fact, is a consequence of Shirota's theorem and the following one [3]: a strongly 0-dimensional, realcompact space \( X \) is \( N \)-compact. Note that a completely regular space \( X \) is strongly 0-dimensional if for any two disjoint zero sets \( F, G \) in \( X \) there exists a clopen set \( U \) of \( X \) such that \( F \subset U \) and \( U \cap G = \emptyset \). Clearly an ultranormal space \( X \) is strongly 0-dimensional and strongly 0-dimensional space is 0-dimensional. It will be interesting to prove the above theorem for 0-dimensional space instead of ultranormal space.)

We shall give an elementary proof of the above theorem here. We begin by stating the already known results.

**LEMMA.** The following are equivalent for a 0-dimensional space \( X \).

a) \( X \) is \( N \)-compact

b) \(([5], ivb)\) for each point \( p_0 \in \beta_D X \setminus X \) there exists a continuous function \( f: \beta_D X \to I \) such that \( f(p_0) = 0 \) and \( f(p) > 0 \) for \( p \in X \).

c) \(([6] \text{ corollary 4.19})\) for each point \( p_0 \in \beta_D X \setminus X \) there exists a continuous function \( f: \beta_D X \to N^* \) such that \( f(p_0) = \infty \) and \( f(p) < \infty \) for \( p \in X \).

**PROOF OF THEOREM.** Let \( X \) be ultranormal and \( \alpha = \{ \mathcal{U} \} \) be a complete uniformity of clopen coverings of \( X \). Let \( B = \bigcap \{ \overline{U} \setminus \mathcal{U} : \mathcal{U} \in \alpha \} \) where \( \overline{U} = \{ U : U \in \mathcal{U} \} \) and \( U \) denotes the closure of \( U \) in \( \beta_D X \). Clearly, \( X \subset B \) and we claim that \( X = B \). If not,
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Let $p_0 \in B \setminus X$. Let $\mathcal{F}$ be the collection of all clopen subsets of $\beta_P X$ about $p_0$. Then $\mathcal{F} \cap X = \{ F \cap X : F \in \mathcal{F} \}$ is a center family of clopen subsets of $X$. For any $U$ in $\alpha$, $p_0 \in U$ for some $U \in \mathcal{U}$. Thus $U \in \mathcal{F}$ and $U = \overline{U} \cap X \in \mathcal{F} \cap X$. Hence $\mathcal{F} \cap X$ is an $\alpha$-cauchy family and since $\alpha$ is complete, $\cap(\mathcal{F} \cap X) \neq \emptyset$. But $\cap(\mathcal{F} \cap X) \subset \cap \mathcal{F} = \{ p_0 \} \subset B \setminus X$. This is a contradiction. Thus $B = X$.

For each point $p_0 \in \beta_P X \setminus X$. Since $X = B$, there exists an $U \in \alpha$ such that $p_0 \in \cup \{ U \mid U \in \mathcal{U} \}$. Since $\alpha$ is a uniformity, there exists a sequence $\{ \mathcal{U}_n \}$ in $\alpha$ such that $\mathcal{U}_1$ star refines $\mathcal{U}$ and $\mathcal{U}_{n+1}$ star refines $\mathcal{U}_n$ for $n = 1, 2, \cdots$. Suppose $\mathcal{U} = \{ U_a : a \in A \}$ where $A$ is assumed to be a well-ordered set of indices. Then as A. H. Stone showed in the proof of (Theorem 1, [9]) there exists a family $F = \{ F_{n,a} : n = 1, 2, \cdots ; a \in A \}$ satisfying the following conditions.

a) $\{ F_{n,a} : n = 1, 2, \cdots ; a \in A \}$ is a closed covering of $X$

b) every element of $\mathcal{U}_{n+3}$ does not intersect two elements of $F_n = \{ F_{n,a} : a \in A \}$ at the same time, and
c) $\text{st}(F_{n,a} \cup F_{n+1}) = \cup \{ U \in U \mathcal{U}_{n+1} : U \cap F_{n,a} \neq \emptyset \} \subset U_a$.

Thus, for each $n = 1, 2, \cdots$, $F_n = \{ F_{n,a} : a \in A \}$ is a discrete system of closed subsets of $X$, so $F = F_1 \cup F_2 \cup F_3 \cup \cdots$ is a $\sigma$-discrete closed covering of $X$ and $F$ is a refinement of $\mathcal{U}$. Hence $p_0 \in F_{n,a}$ for each $n = 1, 2, \cdots$, and $a \in A$.

Let $S_n = \cup F_n$ for each $n = 1, 2, \cdots$. We shall distinguish two cases.

Case 1: $p_0 \notin S_n$ for every $n$. In this case, there exists a continuous function $f_n : \beta_P X \to D$ such that $f_n(p_0) = 0$ and $f_n(p) = 1$ for $p \in S_n$. Setting

$$f(p) = \sum_{n=1}^{\infty} 2^{-n} f_n(p)$$

for $p \in \beta_P X$.

We find that $f : \beta_P X \to I$ is continuous and $f(p_0) = 0$, $f(p) > 0$ for $p \in X$. By Lemma, $X$ is $N$-compact.

Case 2: $p_0 \in S_k$ for some $k$. Since $F_k$ is a discrete system of closed subsets of $X$, $S_k$ is closed in $X$ and hence each member of $F_k$ is clopen in the subspace $S_k$ of $X$ because $S_k - F_{k,a}$ is closed in $X$ for each $a \in A$. Consequently, each member of $F_k$ is clopen in the subspace $S_k \cup \{ p_0 \}$ of $\beta_P X$. Since $p_0 \notin F_{k,a}$ for $a \in A$, $F_{k,a} \cap (S_k \cup \{ p_0 \}) = F_{k,a} \cap (S_k \cup \{ p_0 \}) = F_{k,a}$ so $F_{k,a}$ is closed in $S_k \cup \{ p_0 \}$ for each $a \in A$. Moreover, since $p_0 \in S_k = S_k \setminus \overline{F_{k,a}}$, $p_0 \notin \overline{F_{k,a}}$ and $p_0 \notin \overline{F_{k,a}}$ so $p_0 \in S_k \setminus \overline{F_{k,a}}$ for each $a \in A$. Now $(S_k \cup \{ p_0 \}) \setminus F_{k,a} = (S_k \setminus F_{k,a}) \cup \{ p_0 \} = [(S_k \setminus (S_k \setminus F_{k,a})) \cup (S_k \setminus F_{k,a})] \cup \{ p_0 \}$, $(S_k \setminus F_{k,a}) \cap (S_k \cup \{ p_0 \})$ which is closed in $S_k \cup \{ p_0 \}$ so $F_{k,a}$ is open in $S_k \cup \{ p_0 \}$ for
We consider the collection $F_k \cup \{p_0\}$ as a decomposition space of $S_k \cup \{p_0\}$ and let $\pi$ be the projection of $S_k \cup \{p_0\}$ onto $F_k \cup \{p_0\}$. It follows from the preceding that $\{F_{k,a}\}$ is clopen in $F_k \cup \{p_0\}$ for each $a \in A$. Next, let $M$ be an open set in $F_k \cup \{p_0\}$ and $p_0 \in M$. Then $\pi^{-1}[M] = S_k \cup \{p_0\} \setminus \{F_{k,a} : F_{k,a} \in M\}$ which is closed in $S_k \cup \{p_0\}$ since each $F_{k,a}$ is open in $S_k \cup \{p_0\}$. Hence $M$ is closed in $F_k \cup \{p_0\}$. This proves that $F_k \cup \{p_0\}$ is 0-dimensional and clearly it is also Hausdorff.

If $g : F_k \to D$ then the function $f = g \circ \pi$ is a continuous function defined on $S_k$ into $D$. Since $S_k$ is closed in $X$ and $X$ is ultranormal, $f$ admits a continuous function from $X$ into $D$ and in turn, it admits a continuous extension from $\beta_D X$ into $D$. Hence $f$ admits a continuous extension $f^*$ from $S_k \cup \{p_0\}$ into $D$. Setting $g^*(p) = g(p)$ for $p \in F_k$ and $g^*(p_0) = f^*(p_0)$, we see that the equality $f^* = g^* \circ \pi$ still holds, therefore $g^*$ is a continuous function from $F_k \cup \{p_0\}$ into $D$. In other words, every continuous function $g$ from $F_k$ into $D$ admits a continuous extension from $F_k \cup \{p_0\}$ into $D$ and it means that $p_0$ can be considered as a point from $\beta_D F_k \setminus F_k$.

Since $F_k$ is a discrete system of closed subsets of $X$, according to the property (*) the cardinality of $F_k$ is non-measurable. Moreover, $F_k$ is a discrete subspace of $F_k \cup \{p_0\}$, by Proposition, $F_k$ is $N$-compact. Since $p_0 \in \beta_D F_k \setminus F_k$, by Lemma, there exists a continuous function $g_0$ from $\beta_D F_k$ into $N^k$ such that $g_0(p_0) = \infty$ and $g_0(p) < \infty$ for $p \in F_k$. Define $f_0 : S_k \cup \{p_0\} \to I$ as follows:

$$f_0(p) = \frac{1}{(g_0 \circ \pi)(p)} \quad \text{for } p \in S_k \text{ and } f_0(p_0) = 0$$

We see that $f_0$ is continuous. By theorem 3.1. (2) of [1], the restriction $f_0 | S_k$ has a continuous extension $h$ from $X$ into $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$ such that $h(p) > 0$ for $p \in X$. (Note that if $B = h^{-1}(0) \neq \emptyset$ then $B$ and $S_k$ are disjoint closed subsets of $X$, since $X$ is ultranormal, there is a continuous function $g : X \to D$ such that $g|B = 1$ and $g|S_k = 0$. Let $f_1(p) = \max\{h(p), g(p)\}$ for $p \in X$. Then $f_1$ is a continuous extension of $f_0 | B$ from $X$ into $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$ such that $f_0(p) > 0$ for $p \in X.$) Now, since $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$ is $D$-compact, $h$ admits a continuous extension $h'$ from $\beta_D X$ into $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$. Clearly, $h'(p) > 0$ for $p \in X$ and $h'(p_0) = f_0(p_0) = 0$ ($h'$ and $f_0$ agree on $S_k$ and hence they agree on every point of
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5. $S_k$. This shows that $X$ is $N$-compact.

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REFERENCES


