ON C-CONFORMAL KILLING TENSOR IN A COSYMPLECTIC MANIFOLD

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0. Introduction.

It is well known that a skew symmetric tensor $u_{bc}$ is called a conformal Killing tensor if it satisfies the following equation:

\[(0.1) \quad \nabla_a u_{bc} + \nabla_b u_{ac} = 2\rho_c g_{ab}\]

where $\rho_c$ is a certain vector field. It is a generalization of conformal Killing vector satisfying the Killing-Yano's equation.

On the other hand, Tachibana [2] has defined a conformal Killing tensor in another way. By the definition, a skew symmetric tensor field $u_{bc}$ called a conformal Killing tensor if there exists a vector field $p^a$ satisfying

\[(0.2) \quad \nabla_a u_{bc} + \nabla_b u_{ac} = 2p_c g_{ab} - p_a g_{bc} - p_b g_{ac}\]

Afterward, Yamaguchi [4] has defined a product conformal Killing tensor in a locally product Riemannian manifold and obtained some results. And Chen [1] has defined a F-conformal Killing tensor in Kählerian space and generalized some results.

In this paper we shall define a C-conformal Killing tensor in a cosymplectic manifold and we obtain analogues results to a conformal Killing tensor.

1. Preliminaries.

Let $M$ be a $(2n+1)$-dimensional differentiable manifold with an almost contact metric structure $(\phi^a, \xi^a, \eta_b, g_{ab})$ satisfying

\[(1.1) \quad \phi^a \phi^c_{b} = -\phi^a_{b} + \xi^a \eta_b\]

\[(1.2) \quad \phi^a_{b} \xi^b = 0, \quad \phi^a_{b} \eta_a = 0, \quad \xi^a \eta_a = 1\]

\[(1.3) \quad g_{ab} = \eta_a\]

\[(1.4) \quad g_{cd} \phi^c_{a} \phi^d_{b} = g_{ab} - \eta_a \eta_b\]

If the almost contact structure is normal, then the manifold $M$ is called a normal contact manifold or a Sasakian manifold. An almost contact metric structure is
said to be cosymplectic if it is normal and 2-form $\phi_{ab}=\phi^e_a g_{eb}$ and 1-form $\eta_b$ are both closed. It is known that the cosymplectic is characterized by

\[
\nabla_c \phi^a_b = 0, \quad \nabla_c \eta_b = 0.
\]

Let $R_{abcd}$ and $R_{ab}$ be the curvature tensor and the Ricci tensor respectively. In a cosymplectic manifold, by virtue of (1.5) we have

\[
R_{abcd} \eta^d = 0, \quad R_{ab} \eta^d = 0
\]

If we put

\[
F_{ab} = -\frac{1}{2} R_{abcd} \phi^{cd}
\]

then making use of the Ricci identity for $\phi_{cd}$, we have

\[
R_{abc} \phi_{td} + R_{abd} \phi_{ct} = 0
\]

Contracting $g^{bc}$ to the last equation, we obtain

\[
R_a \phi_{tb} = F_{ab}
\]

from which

\[
F_a \phi_{tb} = -R_{ab}
\]

2. C-conformal Killing tensor.

In this section we shall define a C-conformal Killing tensor in a cosymplectic manifold $M$. For a skew symmetric tensor field $u_{cd}$ if there exists a vector field $p^a$ such that

\[
\nabla_b u_{cd} + \nabla_c u_{bd} = 2p_d g_{bc} - p_b g_{cd} - p_c g_{bd} - 2p_d \eta_b \eta_c + p_b \eta_c \eta_d + p_c \eta_b \eta_d + \eta_{bc} (p_d \phi_{cd} + p_c \phi_{bd})
\]

where we put $\bar{\eta}_c = \phi^e_c \eta^e$ then we call $u_{cd}$ a C-conformal Killing tensor and $p^a$ the associated vector of $u_{cd}$. The associated vector of $u_{cd}$ is given by

\[
p_d = \frac{\nabla^e u_{cd}}{2(n+1)} + \frac{(\nabla^e u_{bc}) \eta^f \eta_d}{2n(n+1)}
\]

and if $p_d$ vanishes identically then $u_{cd}$ is a Killing tensor.

By the definition and (1.4), we have

\[
\bar{p}_c \phi^c = 0, \quad \bar{p}_c \eta^c = 0
\]

\[
p_c \phi^c - \bar{p}_c \phi^c = \lambda^2
\]

where $\lambda = p_c \eta^c$ is a scalar function.
Since we obtain the following formula for any skew symmetric tensor $T_{ab}$, \[ \nabla^a \nabla^b T_{ab} = 0, \]
from (2.2) we get
\[ (\nabla_b p_c) \eta^b \eta^c = (n+1) \nabla^c p_c = 0. \]
Next, we shall seek for differential equations of second order satisfied by $u_{cd}$. If we put
\[ G_{ab} = g_{ab} - \eta_{a} \eta_{b}, \]
then the equation (2.1) becomes
\[ \nabla_a u_{cd} + \nabla_d u_{bc} = 2p_{a} G_{bc} - p_{b} G_{cd} - p_{c} G_{bd} + 3(\bar{p}_{a} \phi_{cd} + \bar{p}_{c} \phi_{bd}). \]
Operating $\nabla_a$ to the last equation, we get
\[ \nabla_a \nabla_b u_{cd} + \nabla_b \nabla_c u_{bd} = 2p_{ab} G_{bc} - p_{ba} G_{cd} - p_{ac} G_{bd} + 3(\bar{p}_{ab} \phi_{cd} + \bar{p}_{ac} \phi_{bd}) \]
where we put
\[ p_{ab} = \nabla_a p_{b}, \quad \bar{p}_{ab} = \nabla_a \bar{p}_{b} = p_{ac} \phi_{bd}. \]
Changing the indices $a, b, c$ cyclically, adding these two equations and subtracting (2.8), we obtain
\[ 2\nabla_a \nabla_b u_{cd} - 2R_{cba} \nabla^{i} u_{di} - R_{bad} \nabla^{i} u_{ct} - R_{acd} \nabla^{i} u_{bd} - R_{bcd} \nabla^{i} u_{at} \]
\[ = 2(p_{ad} G_{bc} + p_{bd} G_{ca} - p_{cd} G_{ab}) - (p_{ab} + p_{ba}) G_{dc} - (p_{ac} + p_{ca}) G_{db} - (p_{bc} + p_{cb}) G_{da} + 3(p_{ab} \phi_{cd} + \bar{p}_{ac} \phi_{bd}) \]
Again, changing the indices $b, c, d$ cyclically and adding these three equations, we have
\[ 2\nabla_a \nabla_b u_{cd} - R_{cba} \nabla^{i} u_{di} - R_{bad} \nabla^{i} u_{ct} - R_{acd} \nabla^{i} u_{bd} - R_{bcd} \nabla^{i} u_{at} \]
\[ = (p_{bd} - p_{db}) G_{ca} + (p_{cb} - p_{bc}) G_{ad} + (p_{cd} - p_{dc}) G_{ab} + (p_{db} - p_{bd}) G_{ba} - 2p_{ac} G_{bd} + 2p_{ad} G_{bc} + (p_{bc} - p_{cb}) \phi_{ad} + (p_{cd} - p_{dc}) \phi_{ab} + (p_{db} - p_{bd}) \phi_{ac} \]
\[ + 2(p_{da} - p_{ad}) \phi_{bc} + 2(p_{ac} - p_{ca}) \phi_{bd} + 2(p_{ab} + p_{ba}) \phi_{cd}, \]
where we have used the following equation
\[ \nabla_a \nabla_b u_{cd} + \nabla_a \nabla_c u_{db} + \nabla_a \nabla_d u_{bc} = 3(\nabla_a \nabla_b u_{cd} + p_{ac} G_{bd} - p_{ad} G_{bc} + \bar{p}_{ad} \phi_{bd} - \bar{p}_{ac} \phi_{bd} + 2 \bar{p}_{ad} \phi_{cd}). \]

3. Integral formula.

In this section we shall prove some integral formula about a tensor field. Let
\( u_{cd} \) be a \( C \)-conformal Killing tensor. Then we obtain

\[
(3.1) \quad \nabla^a \nabla_a u_{cd} - R^{a}_{\ c} u_{da} - R^{a}_{\ cd} u_{ab} =
\]

\[
= -(2n-3)p_{cd} - p_{dc} - 3\delta^a_{\ cd} \delta^b_{\ ab} + 2p_{ad} \eta^a \eta_c + (p_{ca} - p_{ac}) \eta^a \eta_d
\]

\[-3(\delta^a_{\ cd} \delta^b_{\ ab} + 3(\delta^a_{\ ca} - \delta^a_{\ ac}) \delta^b_{\ ab}) \phi^a_d \]

by transvecting (2.9) with \( g^{ab} \).

Now, we shall show that a skew symmetric tensor \( u_{cd} \) satisfying (3.1) is a \( C \)-conformal Killing tensor provided that \( M \) is compact.

If we put

\[
(3.2) \quad U_{bcd} = \nabla_b u_{cd} + \nabla_c u_{bd} - 2p_d G_{bc} + p_b G_{dc} + p_c G_{db} - 3(\delta^a_{\ cd} \delta^b_{\ ab} + 3(\delta^a_{\ ca} - \delta^a_{\ ac}) \delta^b_{\ ab}) \phi^a_d
\]

for a skew symmetric tensor \( u_{cd} \), where \( p_c \) and \( \bar{p}_b \) are given by

\[
p_c = \frac{\nabla^b u_{bc}}{2(n+1)} + \frac{(\nabla^a u_{ab})^b}{2n(n+1)} \eta^a \eta_c
\]

\[
\bar{p}_b = \frac{1}{6(n+1)} (\nabla^b u_{cd} + \nabla^c u_{bd}) \phi^a_d
\]

Simple computations give us the following

\[
(3.3) \quad U_{bcd} U^{bcd} = 2U_{bcd} \nabla^b u^{cd}
\]

\[
(3.4) \quad u^{cd} U_{bcd} = u^{cd} (\nabla^a u_{cd} - R^{a}_{\ c} u_{da} - R^{a}_{\ cd} u_{ab} + (2n-3)p_{cd} + (2n-3)p_{dc} + 2p_{ad} \eta^a \eta_c
\]

\[-(p_{ca} - p_{ac}) \eta^a \eta_d - 3\delta^a_{\ cd} \delta^b_{\ ab} + 3 \delta^a_{\ ca} - \delta^a_{\ ac} \delta^b_{\ ab}) \phi^a_d + \frac{1}{2} U_{bcd} U^{bcd}) \phi^a_d
\]

Substituting (3.3) and (3.4) into

\[
\nabla^b (U_{bcd} u^{cd}) = \nabla^b U_{bcd} u^{cd} + U_{bcd} \nabla^b u^{cd}
\]

Thus we have

**THEOREM 3.1.** In a compact cosymplectic manifold \( M \), the following integral formula is valid for any skew symmetric tensor \( u_{cd} \)

\[
\int_M \left[ u^{cd} (\nabla^a u_{cd} - R^{a}_{\ c} u_{da} - R^{a}_{\ cd} u_{ab} + (2n-3)p_{cd} + (2n-3)p_{dc} - 2p_{ad} \eta^a \eta_c
\]

\[-(p_{ca} - p_{ac}) \eta^a \eta_d - 3\delta^a_{\ cd} \delta^b_{\ ab} + 3 \delta^a_{\ ca} - \delta^a_{\ ac} \delta^b_{\ ab}) \phi^a_d + \frac{1}{2} U_{bcd} U^{bcd}) \phi^a_d \right] d\sigma = 0
\]

where \( d\sigma \) means the volume element of \( M \), \( p_{cd} \) and \( \bar{p}_{cd} \) are given by

\[
p_{cd} = \frac{\nabla^c \nabla^b u_{bd}}{2(n+1)} + \frac{(\nabla^c \nabla^b u_{bd}) \eta^a \eta_d}{2n(n+1)}
\]
Thus we have

**THEOREM 3.2.** In a compact cosymplectic manifold $M$, a necessary and sufficient condition for any skew symmetric $\nu_{ab}$ to be a C-conformal Killing tensor is (3.1).

4. A manifold of constant C-holomorphic sectional curvature.

It has been shown that in a Sasakian manifold or a cosymplectic manifold of constant C-holomorphic sectional curvature $k$, the curvature tensor $R_{abcd}$ has the form

$$R_{abcd} = a(g_{ad}g_{bc} - g_{ac}g_{bd}) + b(\phi_{ad}\phi_{bc} - \phi_{ac}\phi_{bd} - 2\phi_{cd})$$

$$- g_{ad}\eta_{bc} + g_{bd}\eta_{ac} + g_{ac}\eta_{bd},$$

where $a = (k+3)/4$ and $b = (k-1)/4$ in Sasakian manifold, $a = b = k/4$ in cosymplectic manifold. This formula was shown for the Sasakian case by Ogiue and for the cosymplectic case by Blair.

Now we shall show the following

**THEOREM 4.1.** In a cosymplectic manifold of constant C-holomorphic sectional curvature, the covariant derivative $\nabla_c v_d$ of any Killing vector $v_d$ is a C-conformal Killing tensor.

**PROOF.** Let $v_d$ be a Killing vector. Then as is well known we have

$$\nabla_b \nabla_c v_d + R_{abcd} v^a = 0.$$  

Substituting (3.1) into the last equation, we get

$$-v_d \phi_{bc} = -c(v_d g_{bc} - v_c g_{bd} - \bar{v}_c \phi_{bd} + \bar{v}_b \phi_{cd} + 2\bar{v}_d \phi)$$

$$-v_d \eta_{bc} + \lambda g_{bc} \eta + v_d \eta_{bc} + \lambda g_{bd} \eta,$$

where $\bar{v}_d = \phi_d v^a a$ and $\lambda = v^a \eta_a$.

If we put

$$\bar{v}_d = -c(v_d - \lambda \eta_d), \quad \bar{v}_d = \phi_d \eta a - c v_d,$$

then we obtain

$$\nabla_b \nabla_c v_d = \bar{v}_d g_{bc} - \bar{v}_c g_{bd} - \bar{v}_b \phi_{bd} - \bar{v}_d \phi_{cd} + 2\bar{v}_b \phi_{cd}$$

$$- \bar{v}_d \eta_{bc} - \bar{v}_c \eta_{bd} - \bar{v}_d \eta_{bc}.$$

Changing the indices $b$ and $c$, adding these two equations, we have
This equation shows that $\nabla_p v_d$ is a $C$-conformal Killing tensor.

We know the converse of Theorem 3.1 is valid as follows.

**THEOREM 4.2.** In a cosymplectic manifold $M$, if Lie algebra of all Killing vectors $v_d$ is transitive and the covariant derivative $\nabla_p v_d$ of any Killing vector $v_d$ is a $C$-conformal Killing tensor, then $M$ is a manifold of constant $C$-holomorphic sectional curvature.

**PROOF.** Taking $u_{cd} = \nabla_c v_d$ in (2.7) and by making use of (3.2), we have

$$-(R_{abcd} + R_{acbd})u^a$$

$$= (2p_d G_{bc} - p_b G_{cd} - p_c G_{bd}) + 3(p_{b\phi d} + p_{c\phi bd})$$

Transvecting (4.4) with $g^{bc}$ and $\phi^{cd}$ respectively, we have

$$-R_{ad} u^d = 2(n+1)p_d$$

(4.5)

$$-F_{ab} u^a = 2(n+1)p_b$$

(4.6)

by virtue of (1.5).

Substituting (4.5) and (4.6) into (4.4), we have

$$-(R_{abcd} + R_{acbd})u^a = rac{1}{2(n+1)}(2R_{ad} G_{bc} - R_{ab} G_{cd} - R_{ac} G_{bd}$$

$$+ 3(F_{ab\phi cd} + F_{ac\phi bd})u^a)$$

Since the last equation holds for any vector $u^a$, we obtain

$$R_{abcd} + R_{acbd} = \frac{1}{2(n+1)}(2R_{ad} G_{bc} - R_{ab} G_{cd} - R_{ac} G_{bd}$$

$$+ 3(F_{ab\phi cd} + F_{ac\phi bd}))$$

(4.7)

Transvecting (4.7) with $g^{ad}$ and taking account of (1.8), we have

$$R_{bc} = \frac{1}{2n}RG_{bc}$$

(4.8)

Substituting the last equation into (4.7), we have

$$R_{abcd} + R_{acbd} = \frac{R}{2n(n+1)}(2G_{ad} G_{bc} - G_{ab} G_{cd} - G_{ac} G_{bd}$$

$$+ 3(\phi_{ab\phi cd} + \phi_{ac\phi bd}))$$

(4.9)

Interchanging indices $b, c, d$ in (4.9) as $b \rightarrow c \rightarrow d \rightarrow b$ and then substracting what follows from (4.9), we have
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\[ R_{acbd} = \frac{R}{2n(n+1)} (g_{ac} g_{bd} - g_{ad} g_{bc} + \phi_{ac} \phi_{bd} - \phi_{ad} \phi_{bc} - 2 \phi_{ac} \phi_{bd}) \]

taking account of \( G_{bc} = g_{bc} - \eta_b \eta_c \), the last equation becomes

\[(4.10) \quad R_{acbd} = \frac{R}{2n(n+1)} (g_{ad} g_{bc} - g_{ac} g_{bd} + \phi_{ad} \phi_{bc} - \phi_{ac} \phi_{bd} - 2 \phi_{ad} \phi_{bc} - g_{cd} \eta_b \eta_c - g_{bc} \eta_c \eta_d + g_{ad} \eta_c \eta_d - g_{cd} \eta_a \eta_b) \]

Thus the proof is complete.

Let us assume that \( c \neq 0 \). If we put

\[(4.11) \quad q_{cd} = u_{cd} + \frac{1}{c} \nabla_c \phi_d \]

then by virtue of \((4.3)\) and \((2.1)\), it follows that

\[ \nabla_b q_{cd} + \nabla_c q_{bd} = 0, \]

which means \( q_{cd} \) is a Killing tensor. Consequently, a C-conformal tensor \( u_{cd} \) is decomposed in the form

\[(4.12) \quad u_{cd} = q_{cd} + p_{cd} \]

where \( q_{cd} \) is a Killing tensor and \( p_{cd} = -\frac{1}{a} \nabla_c \phi_d \) is a closed C-conformal Killing tensor. Thus we have

**Theorem 4.3.** In a cosymplectic manifold of constant C-holomorphic sectional curvature \( a = R/2n(n+1) \neq 0 \), a C-conformal Killing tensor \( u_{cd} \) is decomposed in the form

\[ u_{cd} = q_{cd} + p_{cd} \]

where \( q_{cd} \) is a Killing tensor and \( p_{cd} \) is a closed C-conformal Killing tensor. In this case \( p_{cd} \) is the form

\[ p_{cd} = -\frac{1}{a} \nabla_c \phi_d \]

where \( \phi_d \) is the associated vector of \( u_{cd} \). Conversely if \( q_{cd} \) is a Killing tensor and \( p_c \) is a Killing vector, then \( u_{cd} \) given by \((4.12)\) is a C-conformal Killing tensor.

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**BIBLIOGRAPHY**


