ON A SPACE OF CONSTANT CURVATURE WITH \((f, g, u, v, \lambda)\)-STRUCTURE

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§ 1. Introduction

In previous paper [3], the present authors proved the theorem:

**Theorem 0.1.** Let \(M\) be a complete quasi-normal \((f, g, u, v, \lambda)\)-structure satisfying one of the following conditions:

\[
\mathcal{L}_U g = 2\alpha \lambda g, \quad df = 2\beta \omega,
\]

\[
\mathcal{L}_V g = 2\gamma \lambda g, \quad dv = 2\delta \omega,
\]

\(\alpha, \beta, \gamma\) and \(\delta\) being non-zero functions and \(\mathcal{L}_U\) denoting the operator of Lie differentiation with respect to the vector field \(U\). If \(\lambda(1-\lambda^2)\) is almost everywhere non-zero function and \(\dim M > 2\), then \(M\) is isometric with an even-dimensional sphere \(S^{2n}\).

The main purpose of the present paper is to prove the following theorem:

**Theorem A.** Let \(M\) (\(\dim M > 2\)) be a complete Riemannian space of constant curvature with \((f, g, u, v, \lambda)\)-structure. If \(M\) satisfies

\[
(0.1) \quad \nabla_{\partial_i} \phi = \phi f_{j\mu} \delta_{ij},
\]

\(\phi\) being non-zero differentiable function, and \(\lambda(1-\lambda^2)\) is almost everywhere non-zero function, then \(M\) is isometric with an even-dimensional sphere.

§ 2. A space of constant curvature with \((f, g, u, v, \lambda)\)-structure

Let \(M\) be a \(2n\)-dimensional differentiable manifold of class \(C^\infty\) with an \((f, g, u, v, \lambda)\)-structure, that is, a tensor field \(f\) of type \((1,1)\), a positive definite Riemannian metric \(g\), two 1-forms \(u_i\) and \(v_i\) (or two vector fields associated with them), and a function \(\lambda\) satisfying

\[
(2.1) \quad f^j_i f^h_i = -\delta^h_j + u^h \mu^h + v^h \nu^h,
\]

\[
(2.2) \quad f^i_i u^i = -\lambda v^i, \quad u_i f^i_j = \lambda v^i,
\]

\[
(2.3) \quad f^i_i v^i = \lambda u^i, \quad v_i f^i_j = -\lambda u^i.
\]
(2.4)  \[ u_{ij}' = 1 - \lambda^2, \quad u_{ij} = 0, \]

(2.5)  \[ v_{ij}' = 1 - \lambda^2, \quad v_{ij} = 0, \]

(2.6)  \[ g_{ij} f_i^j f_j^i = g_{ji} - u_{ij} - v_{ij}. \]

We put

(2.7)  \[ S_{ji}^h = f_i^j \nabla_i f_j^h - f_i^j \nabla_j f_j^h - (\nabla_i f_i^j - \nabla_j f_j^j) f_j^h + u_{ij} \rho^h + v_{ij} \rho^h, \]

where

(2.8)  \[ u_{ij} = \nabla_i u_j - \nabla_j u_i, \quad v_{ij} = \nabla_i v_j - \nabla_j v_i, \]

\( \nabla_j \) denoting the operator of covariant differentiation with respect to the Riemannian connection. If the tensor \( S_{ji}^h \) vanishes, the \((f, g, u, v, \lambda)\) -structure is said to be normal [1].

Transvecting (2.7) with \( \rho \) and using (2.1)~(2.5), we can prove

(2.9)  \[ S_{ji}^h \rho^h = \nabla_j f_i^h + \nabla_j f_{ij} + \rho \nabla_i f_i^h + \rho \nabla_j f_{ij}, \]

We now put

(2.10)  \[ f_{jih} = \nabla_j f_i^h + \nabla_j f_{ij} + \nabla_i f_{ji}. \]

If \( S_{ji}^h - (f_i^j f_i^h - f_i^j f_{ij}^h) = 0 \), then the structure is said to be quasi-normal [2].

Moreover, if a space has the curvature tensor of the form

(2.11)  \[ R_{klij} = c (g_{khi} g_{lij} - g_{kli} g_{jih}) \quad (c = \text{constant}), \]

then it is said to be a space of constant curvature.

In the sequel we assume \( \lambda(1 - \lambda^2) \) is almost everywhere non-zero function on \( M \).

§ 3. Proof of Theorem A

Let \( M \) be a Riemannian space of constant curvature with \((f, g, u, v, \lambda)\) -structure. If \( M \) satisfies (0.1), then by transvecting (0.1) with \( \rho \) we find

(3.1)  \[ \nabla_i \rho^h = \phi \nu_i^j, \]

Differentiating (3.1) covariantly, we get

\[ \nabla_k \nabla_i \rho^h = (\nabla_k \rho^h) u_i^j + \phi \nabla_k \nabla_i \rho^h, \]

from which,

(3.2)  \[ 0 = (\nabla_k \rho^h) u_i^j - (\nabla_i \rho^h) u_k^h + \phi (\nabla_k \nabla_j^h - \nabla_j \nabla_k^h). \]
We have from (0.1)
\( \nabla p_i - \nabla v_j = 2\phi f_{ji} \)
which implies
\[
0 = (\nabla \phi)f_{ji} + (\nabla \phi)f_{ih} + (\nabla \phi)f_{kj} + \phi f_{ji}.
\]
Since \( \nabla_k \nabla_{p_i} = (\nabla \phi)f_{ji} + \phi \nabla_k f_{ji} \),
we obtain
\[
\nabla_k \nabla v_i - \nabla_j \nabla v_i = (\nabla \phi)f_{ji} - (\nabla \phi)f_{hk} + \phi (\nabla_k f_{ji} - \nabla_j f_{ki}),
\]
from which, using (2.10), (2.11) and (3.4),
\[
-(\nabla \phi)(\nabla f_{ki}) = - (\nabla \phi)f_{ji} - \phi \nabla f_{jk}.
\]
Transvecting (3.5) with \( f^{kj} \) and taking account of (2.1) and (2.3), we find
\[
-c(\lambda u_i + \lambda u_j) = - (\nabla \phi)(2n - 2(1 - \lambda^2)) - \phi \nabla_i n - (1 - \lambda^2),
\]
or, using (3.1),
\[
(n - (1 - \lambda^2)) \nabla \phi = - \lambda(\phi^2 - c)u_i.
\]
Using (3.2) and (3.6) and taking account of \( n - (1 - \lambda^2) \neq 0 \), we have
\[
(\nabla \phi)(\nabla f_{ki}) = 0.
\]
Owing to (2.7), (3.3), (3.5) and (3.7), we find
\[
\phi S_{kji} = -f_{jk}(\nabla \phi)_{f_{ji}} + f_{jk}(\nabla \phi)_{f_{ki}} + (\nabla \phi)(- g_{ji} + u_i v_j) - (\nabla \phi)(- g_{ki} + u_k v_j + v^2) + 2(\phi^2 - c)f_{ki}.\]
Transvecting (3.8) with \( v^i \) and using (2.2), (2.3), (2.4) and (2.5), we have
\[
\phi S_{kji} = -f_{jk}(\nabla \phi)_{f_{ji}} - \lambda f_{jk}(\nabla \phi)_{f_{ki}} + (\nabla \phi)(- g_{ji} + u_i v_j) - (\nabla \phi)(- g_{ki} + u_k v_j + v^2) + 2(\phi^2 - c)(1 - \lambda^2)f_{ki}.
\]
Multiplying (2.9) by \( \phi \) and substituting (3.1), (3.3) and (3.7) in the equation obtained, we have \( \phi S_{kji} = 0 \). Thus, from (3.6) and (3.9), we get
\[
0 = - \lambda^2(\phi^2 - c)u_k v_j + \lambda^3(\phi^2 - c)u_i v_k + (1 - \lambda^2)(n - (1 - \lambda^2))(\phi^2 - c)f_{jk}.
\]
from which, transvecting \( u^i s^k \),
\[
\phi^2 - c = 0
\]
which means that \( \phi \) is constant. Consequently (3.4) and (3.8) can be respectively written as \( f_{kji} = 0 \) and \( S_{kji} = 0 \). Hence our structure is quasi-normal. Combining Theorem 0.1, we get the result.
BIBLIOGRAPHY

