ON AN APPLICATION OF THE STEREOGRAPHIC PROJECTION TO $CP^n$

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§ 0. Introduction. Let $M^n$ (resp. $M^{m}$) be a Riemannian space of metric $g$ (resp. $g'$) and $\phi$ a diffeomorphism from $M^n$ to $M^{m}$. If $\phi$ maps any geodesic in $M^n$ to a geodesic in $M^{m}$, it is called projective. The projective curvature tensor $W$ is preserved by any projective map $\phi$, i.e. we have $W=\phi^*(W')$. For a diffeomorphism $\phi$ if there exists a scalar function $\sigma$ such that $\phi^*(g')=e^{2\sigma} g$, we call $\phi$ conformal. The conformal curvature tensor $C$ is preserved by any conformal map.

Let $K^n$ be a Kählerian space and $\Gamma^\lambda_{\mu \nu}$ Christoffel symbols with respect to a local coordinate $\{z^\lambda\}$. A curve $c$ in $K^n$ is called a holomorphically planar (or $H$-plane) curve if $c$ is represented as $z^\lambda=z^\lambda(t)$ and satisfy

$$\frac{d^2 z^\lambda}{dt^2}+\Gamma^\lambda_{\mu \nu} \frac{dz^\mu}{dt} \frac{dz^\nu}{dt}=\alpha \frac{dz^\nu}{dt},$$

where $\alpha$ is a complex-valued function of $t$, [9], [11].

Consider a diffeomorphism $\phi$ of $K^n$ to another $K^{m}$. An $H$-projective map is a diffeomorphism which maps any $H$-plane curve to an $H$-plane curve. A holomorphic $\phi$ is $H$-projective if and only if there exists a self-adjoint vector $\rho_\lambda$ such that

$$\phi^*(\Gamma^\lambda_{\mu \nu})=\Gamma^\lambda_{\mu \nu}+\rho_\mu \delta^\lambda_{\nu}+\rho_\nu \delta^\lambda_{\mu},$$

where $\Gamma^\lambda_{\mu \nu}$ mean the Christoffel symbols of $K^{m}$.

We have known a lot of theorems about $H$-projective maps which correspond to ones of projective maps, [9], [11]. Especially, corresponding to $W$, the $H$-projective curvature tensor $P$ has been shown as an invariant under $H$-projective maps, [11].

Now it would be natural to ask for a diffeomorphism $\phi$ of $K^n$ to $K^{m}$ having the property $x$ such that

projective: $H$-projective=conformal: $x$. 
It seems that the Bochner curvature tensor $K$ of $\mathbb{K}^n$ gives support to the existence of $x$. Because we may consider a symbolical relation

$$\mathcal{W} : P = C : K$$

to be valid among the defining equations of these tensors, [4], [12]. Actually, some theorems for Riemannian spaces of $C = 0$ have been generalized to for Kählerian spaces of $K = 0$, [4], [5], [10]. Thus $K$ would be preserved by $\phi$ of property $x$.

On the other hand, let $S^n$ be the unit sphere in the Euclidean $(n+1)$-space $E^{n+1}$. If we denote by $\Phi$ the central (or stereographic) projection from $S^n$ to an $E^n$ (selected suitably), $\Phi$ is a projective (or conformal) map. Hence, for any projective (or conformal) local transformation $\phi$ in $E^n$, $\Phi^{-1} \circ \phi \circ \Phi$ is projective (or conformal) on $S^n$.

The complex projective space $CP^n$ is one of typical examples of Kählerian spaces, and is a quotient space of $S^{2m+1}$ by a certain equivalence relation, [1]. Making use of the central (or stereographic) projection $\Phi$ of $S^{2m+1}$ to $E^{2m+1}$, an equivalence relation can be introduced in $E^{2m+1}$ and the induced map $\Phi$ is defined so that the commutativity holds in the diagram:

$$\begin{array}{ccc}
S^{2m+1} & \xrightarrow{\Phi} & E^{2m+1} \\
\downarrow & & \downarrow \\
CP^n = S^{2m+1}/\sim & \xrightarrow{\Phi} & E^{2m} = E^{2m+1}/\sim
\end{array}$$

For the central projection $\Phi$, we may expect $\Phi$ to be H-projective. $\Phi$ would have the property $x$ for the stereographic $\Phi$.

The purpose of this paper is mainly to discuss on $\Phi$ for the stereographic projection $\Phi$. $x$ is not fixed yet in this paper and still remains as a question.

Throughout the paper we shall agree with the following conventions.

(1) The ranges of indices.

- $A, B, C, \ldots = 1, \ldots, 2m+2$.
- $a, b, c, \ldots = 1, \ldots, m+1$.
- $\lambda, \mu, \nu, \ldots = 1, \ldots, m$.
- $j, k, \ell, \ldots = 1, \ldots, m, 1^*, \ldots, m^*$.

(II) $\Delta = m+1, \Delta^* = 2m+2$.

(III) Indices with $*$. For real coordinates—say $\{y^A\}$, $Y^a = y^a + (m+1)$,

For complex coordinates—say $\{z^\lambda\}$.
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\(z^* = \overline{z}\) (complex conjugate), \(\{z^b, \overline{z}^b\}\).

(W) The summation convention. For examples,
\[Y^A Y^A = Y^1 Y^1 + \ldots + Y^{2m+1} Y^{2m+1}.\]
\[z^2 dz^2 = z^1 dz^1 + \ldots + z^{m+1} dz^{m+1}.\]
\[u^1 Y^1 = u^1 Y^1 + \ldots + u^m Y^m.\]

§ 1. The canonical metric of \(\mathbb{CP}^m\). Consider the Euclidean space \(\mathbb{E}^{2m+2}\) of dimension \(2m+2\), \(m \geq 1\), and we denote by \(\{Y^a\}\) a fixed orthogonal coordinate system of origin \(O\). Let \(\{w^a\}\) be the complex coordinate system in \(\mathbb{E}^{2m+2}\) associated to \(\{Y^a\}\):
\[w^a = Y^a + i Y^{a*}.\]

\(S^{2m+1}\) means the unit hypersphere of center \(O\) defined by
\[Y^A Y^A = w^a w^{a*} = 1.\]

Let \(\langle w^a \rangle\) and \(\langle w^a \rangle\) be points on \(S^{2m+1}\). If there exists a \(\theta\) such that
\[w^a = e^{i\theta} w^a, \quad 0 \leq \theta \leq 2\pi.\]

then we shall say \(\langle w^a \rangle\) to be equivalent to \(\langle w^a \rangle\), and represent it by \(\langle w^a \rangle \sim \langle w^a \rangle\).

As this relation \(\sim\) clearly satisfies the three conditions of equivalence relation, \(S^{2m+1}\) is classified into the set
\[\mathbb{CP}^m = S^{2m+1} / \sim\]

of the equivalence classes. \(\mathbb{CP}^m\) is called the complex projective space. It is an \(m\) (complex-) dimensional complex manifold with the natural structure. In fact, the natural local coordinates \(\{V_b, z^b_\lambda\}, \ b=1, \ldots, m+1\), of \(\mathbb{CP}^m\) is introduced as follows: For each \(b\), \(U_b\) and \(V_b\) denote the sets given by
\[U_b = \{w^a \in S^{2m+1} | w^b \neq 0\}, \ V_b = U_b / \sim,\]

and let \(\{z^b_\lambda\}\) on \(V_b\)
\[z^b_\lambda = \frac{w^\lambda\_b}{w^b}, \quad \lambda = 1, \ldots, b-1,\]
\[z^b_\mu = \frac{w^{\mu+1}b}{w^b}, \quad \mu = b, \ldots, m.\]

We shall consider a geometrical meaning of (1.1). In terms of \(\{Y^A\}\), (1.1) is written as
(1.2) \[ \begin{align*} y'^a &= y^a \cos \theta - y^{a*} \sin \theta, \\ y'^{a*} &= y^{a*} \cos \theta + y^a \sin \theta. \end{align*} \]

Let \( Y \) and \( \vec{Y} \) be the vectors in \( E^{2m+2} \) defined by
\[ Y = \begin{pmatrix} y^a \\ y^{a*} \end{pmatrix}, \quad \vec{Y} = JY = \begin{pmatrix} -y^{a*} \\ y^a \end{pmatrix}. \]

Here, \( J \) means the natural almost complex structure in \( E^{2m+2} \), i.e., the matrix \( J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \), \( O_m \) and \( I_m \) being the zero and the unit matrix respectively. As the vector \( Y \) at \( (Y^A) \) on \( S^{2m+1} \) is regarded as the unit normal vector to \( S^{2m+1} \) at the point, \( \vec{Y} \) is tangent to \( S^{2m+1} \) at \( (Y^A) \). The set of \( Y \) at each point \( (Y^A) \) on \( S^{2m+1} \) constitutes a vector field \( \vec{Y} \) over \( S^{2m+1} \), and it is known that \( \vec{Y} \) is a unit Killing vector in \( S^{2m+1} \) with the natural structure of a space of constant curvature. \( \vec{Y} \) is called a Sasakian structure on \( S^{2m+1} \). The equation (1.2) is written as
\[ Y' = Y \cos \theta + \vec{Y} \sin \theta. \]

Thus, the equivalence class of a point \( (Y^A) \) is a great circle which is an integral curve of the Sasakian structure \( \vec{Y} \) because of
\[ \left( \frac{dY'}{d\theta} \right)_{\theta=0} = \vec{Y}. \]

It is known that \( S^{2m+1} \) is a fibre bundle over \( CP^m \) with fibre \( S^1 \), called Hopf fibering.

Henceforward, our discussions will be done only in
\[ U_\Delta = \{ (w^\alpha) | w^\Delta \neq 0 \} \quad \text{and} \quad V_\Delta = U_\Delta / \sim. \]

The canonical (Kähler) metric of \( CP^m \) is defined in \( V_\Delta \) by
\[ (1.3) \quad ds^2_z = \frac{2}{f} (f^2 |z^\lambda \bar{z}^{\lambda*} - |z^\lambda| z^{\lambda*}|^2), \]
where
\[ z^\lambda = \frac{w^\Delta}{w^\lambda}. \]

\[ (1.4) \quad f = \sqrt{1 + u}, \quad u = z^\lambda \bar{z}^{\lambda*}. \]

The metric (1.3) is written in the form
\[ ds^2_z = g_{j\bar{h}} dz^j d\bar{z}^h = 2g_{2u} d\bar{z}^\lambda dz^{\lambda*}, \]
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with $g_{\lambda\mu} = g^{\lambda\mu} = 0$ and

$$g_{\lambda\mu} = \frac{1}{f^2} (r^2 \partial_{\lambda\mu} - z^{\lambda\mu}).$$

$(g^{jk})$ is given by $g^{\lambda\mu} = g^{\lambda\mu} = 0$ and

$$g^{\lambda\mu} = \frac{1}{2} (\partial_{\lambda\mu} + z^{\lambda\mu}).$$

The Christoffel symbols $\Gamma^h_{jl}$ are all zero except

$$\Gamma^h_{\mu\nu} = -\frac{1}{f^2} (\partial_{\mu}^{\lambda} \partial^{\nu}_{\lambda} + \partial_{\nu}^{\lambda} \partial^{\lambda}_{\lambda})$$

and their complex conjugates.

The non-vanishing components of the curvature tensor $R^h_{jkl}$ are ones which follow by the algebraic identities about $R^h_{jkl}$ from

$$R^a_{\mu\nu\sigma} = - (\partial_{\mu}^{\lambda} \partial^{\nu}_{\lambda} + \partial_{\nu}^{\lambda} \partial^{\lambda}_{\lambda})$$

and their complex conjugates. [12].

It is $\mathbb{CP}^m$ with this metric what we shall denote by $\mathbb{CP}^m$ in the rest of this paper. $\mathbb{CP}^m$ is a space of constant holomorphic curvature.

§ 2. The central projection $\bar{\Phi}$. Denoting the north pole of $S^{2m+1}$ by $(y_0^A)${

$y^A_0 = 1$, $y_0^A = 0$, $A \neq \Delta$.

we consider the tangent hyperplane

$$E^{2m+1} : Y^\Delta = 1$$

of $S^{2m+1}$ at $(y_0^A)$. Let

$$\Phi : S^{2m+1} - S_\Delta \rightarrow E^{2m+1}$$

be the central projection, where $S_\Delta^{2m}$ denotes the equator $Y^\Delta = 0$ on $S^{2m+1}$.

Consider a point $\bar{P} \in \mathbb{CP}^m$. If $\bar{P} \in V_\Delta$, the equivalence class $\bar{P}$ contains a point $P(y^A) \in S^{2m+1}$ such that $y^\Delta \neq 0$. As the equation of line $OP$ is

$$Y^A = ty^A$$

the coordinates of $\Phi(P)$ are

$$u^A = \frac{y^A}{y^\Delta}.$$

If $P'(y'^A)$ be a point equivalent to $P$, the coordinates of $\Phi(P')$ are $u'^A = y'^A/y'^\Delta$ which are written by virtue of (1.2) as
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\[ u^\lambda = (y^\lambda \cos \theta - y^\lambda^* \sin \theta)/y^\Lambda, \]
\[ u^\lambda^* = (y^\lambda \sin \theta + y^\lambda^* \cos \theta)/y^\Lambda, \]
\[ u^\Lambda^* = (y^\Lambda \sin \theta + y^\Lambda^* \cos \theta)/y^\Lambda \]
on $E^{2m+1}$, where
\[ y^\Lambda = y^\Lambda \cos \theta - y^\Lambda^* \sin \theta. \]

Since $\Phi$ is the central projection and $\bar{P}$ is a great circle of $S^{2m+1}$, the equation (2.1) represents a line on $E^{2m+1}$ with parameter $\theta$. On the other hand, it is evident geometrically that different two points of $V_\Delta$ go to two non-intersecting lines on $E^{2m+1}$. Thus $\Phi$ induces a map from $V_\Delta$ into $E^{2m+1}/\sim$, where $\sim$ means the equivalence relation induced by $\Phi$.

Next we shall consider
\[ (2.2) \quad E^{2m} : \quad Y^\Lambda^* = 0 \]
on $E^{2m+1}$, and find the point (denoted by $\tilde{\phi}(\bar{P})$) at where the line (2.1) meets with $E^{2m}$.

At the point, we have from (2.1) and (2.2)
\[ y^\Lambda \sin \theta + y^\Lambda^* \cos \theta = 0. \]
Substituting these values of $\theta$ into (2.1), we can get
\[ (2.3) \quad u^\lambda = \alpha (y^\lambda \lambda^\lambda + y^\lambda^* \lambda^*), \]
\[ u^\lambda^* = \alpha (y^\lambda^\lambda \lambda^* - y^\lambda^* \lambda^*), \]
where
\[ (2.4) \quad 1/\alpha = (y^\Lambda)^2 + (y^\Lambda^*)^2 = w^\Lambda \ w^\Lambda^*. \]
Thus we obtain a map
\[ \tilde{\phi} : V_\Delta \longrightarrow E^{2m} \]
which brings $\bar{P}$ to $\tilde{\phi}(\bar{P})$ given by (2.3).

We shall represent $\tilde{\phi}$ in terms of local coordinates
\[ (2.5) \quad z^\lambda = \frac{w^\lambda}{w^\Lambda} \quad \text{in } V_\Delta \]
and the complex coordinate in $E^{2m}$ given by
\[ (2.6) \quad \alpha^\lambda = u^\lambda^* + i u^\lambda \]
which is the restriction of $\{w^\sigma\}$ in $E^{2m+2}$ to $E^{2m}$.

Substituting (2.3) into (2.6) and taking account of (2.4) and (2.5), we obtain $\alpha^\lambda = z^\lambda$. Hence we know that $\tilde{\phi}$ is given by
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and consequently $\varphi$ is 1-1 holomorphic.

$\varphi$ will be called the central projection of $CP^m$ to $E^{2m}$.

Now consider the canonical (Kähler) metric

$$ds^2 = 2d\alpha^2 d\alpha^*$$

on $E^{2m}$, then the induced metric on $V_\Lambda$ by $\varphi$ is

(2.7) $$\varphi^*(ds^2) = 2dz^2 d\lambda^*.$$

The Christoffel symbols $\varphi^*(\Gamma^\lambda_{\mu\nu})$ of (2.7) being all zero, we have

$$\varphi^*(\Gamma^\lambda_{\mu\nu}) = \Gamma^\lambda_{\mu\nu} + \frac{1}{2f^2}(\delta^\lambda_{\mu\nu} + \delta^\lambda_{\mu\nu})$$

by taking account of (1.5). Consequently, $\varphi$ is an $H$-projective transformation.

§ 3. The stereographic projection $\Psi$. Let $P_1(y^A_1)$ be the south pole of $S^{2m+1}$ given by

$$y^A_1 = -1, \quad y^A = 0, \quad A \neq \Delta.$$

Consider the stereographic projection

$$\Psi : S^{2m+1} - \{P_1\} \longrightarrow E^{2m+1}$$

where

$$E^{2m+1} : \quad y^\Lambda = 0.$$

For a point $P(y^A) \neq P_1$, the line $P_1P$ is given by

$$Y^A = y^A_1 + t(y^A - y^A_1), \quad t : \text{real},$$

or equivalently

$$Y^A = ty^A, \quad A \neq \Delta,$$

$$Y^\Lambda = -1 + t(y^\Lambda + 1).$$

Therefore the value of $t$ is

$$t = \frac{1}{y^\Lambda + 1}$$

at the point $\Psi(P)$, the intersecting point of line $P_1P$ with $E^{2m+1}$.

Thus the coordinates of $\Psi(P)$ on $E^{2m+1}$ are

$$u^A = y^A/(y^\Lambda + 1), \quad A \neq \Delta.$$

For a point $P'(y'^A)$ equivalent to $P$, $\Psi(P')$ has coordinates

$$u^A = y'^A/(y'^\Lambda + 1), \quad A \neq \Delta.$$
and hence taking account of (1.2)

\[ u^\lambda = \beta(y^\lambda \cos \theta - y^{\lambda*} \sin \theta) \]

\[ u^{\Lambda*} = \beta(y^{\Lambda*} \cos \theta + y^\Lambda \sin \theta) \]

(3.1)

where

\[ \frac{1}{\beta} = y^{\lambda*} + 1 = y^\Lambda \cos \theta - y^{\Lambda*} \sin \theta + 1. \]

As \( \Psi \) is a conformal map, the class \( \bar{P} \) of \( P \) is mapped by \( \Psi \) to a circle or a line given by (3.1).

Denoting

\[ E^{2m} : \quad \gamma^\Lambda* = 0, \]

we shall find the intersecting points of (3.1) with \( E^{2m} \). At the points \( \theta \) takes the values such that

(3.3) \[ y^\Lambda* \cos \theta + y^\Lambda \sin \theta = 0, \]

and hence we have

(3.4) \[ \cos \theta = \pm y^\Lambda / \sqrt{(y^\Lambda)^2 + (y^{\Lambda*})^2} = \pm y^\Lambda / \sqrt{w^\Lambda w^{\Lambda*}} \]

if \( \bar{P} \) is in \( V_\Delta \).

In the following we shall adopt + sign in (3.4) and denote by \( \bar{\psi}(\bar{P}) \) the point corresponding to that value of \( \theta \).

The coordinates of \( \bar{\psi}(\bar{P}) \) are

(3.5)

\[ u^\lambda = \gamma (y^\lambda y^\lambda + y^{\Lambda*} y^{\lambda*}), \]
\[ u^{\Lambda*} = \gamma (y^\Lambda y^{\Lambda*} - y^{\lambda*} y^\lambda) \]

by (3.1), (3.2), (3.3) and (3.4), where

(3.6) \[ \gamma = 1/(w^\Lambda w^{\Lambda*} + \sqrt{w^\Lambda w^{\Lambda*}}). \]

Next we shall represent \( \bar{\psi} \) in terms of local coordinates \( \{z^\lambda\} \) in \( V_\Delta \) and

(3.7) \[ \alpha^\lambda = u^\lambda + i u^{\Lambda*} \]

in \( E^{2m} \). Substituting (3.5) into (3.7) and taking account of

\[ z^\lambda = \frac{w^\lambda}{w^\Lambda} = \frac{-y^\lambda + iy^{\Lambda*}}{y^\Lambda + iy^{\lambda*}} \]

and (3.6), we can get

(3.8) \[ \alpha^\lambda = z^\lambda / (f+1), \]

where \( f \) means the one in (1.4), i.e.,

\[ = \sqrt{1+u}, \quad u = z^\lambda z^{\Lambda*} \]
As we have
\[ (3.9) \quad \alpha^\lambda \alpha^{\lambda^*} = \frac{f-1}{f+1}, \quad \frac{f}{1-\alpha^\lambda \alpha^{\lambda^*}}. \]
the equation (3.8) is solved for \( z^\lambda \) as
\[ (3.10) \quad z^\lambda = \frac{2}{1-\alpha^\lambda \alpha^{\lambda^*}}. \]
It follows from (3.9)
\[ (3.11) \quad |\alpha^\lambda \alpha^{\lambda^*}| < 1, \]
and hence we get a diffeomorphism
\[ \psi : V_\Lambda \longrightarrow B^{2m} \]
given by (3.8), where \( B^{2m} \) is the domain in \( B^{2m} \) satisfying (3.11).
\( \psi \) will be called the **stereographic projection** of \( CP^m \) to \( E^{2m} \).

**§ 4. The induced metric.** We shall calculate the induced metric of the canonical (Kähler) metric
\[ ds_\alpha^2 = 2d\alpha^\lambda d\alpha^{\lambda^*} \]
of \( E^{2m} \) by the stereographic projection
\[ \psi : z^\lambda \longrightarrow \alpha^\lambda = z^\lambda/(f+1). \]
As we have
\[ d\alpha^\lambda = \frac{1}{f+1} dz^\lambda - \frac{1}{(f+1)^2} z^\lambda df, \]
the induced metric is given by
\[ (4.1) \quad \psi^*(ds_\alpha^2) = \frac{2}{(f+1)^2} \{ dx^\lambda dx^{\lambda^*} - (df)^2 \}. \]
On the other hand, the metric \( ds_z^2 \) of \( CP^m \) being (1.3), we have
\[ 2dz^\lambda dz^{\lambda^*} = f^2 ds_z^2 + \frac{2}{f^2} |z^\lambda dz^{\lambda^*}|^2. \]
If we substitute the last equation into (4.1) and take account of
\[ df = (z^{\lambda^*} dz^\lambda + z^\lambda dz^{\lambda^*})/2f, \]
\[ |z^\lambda dz^{\lambda^*}|^2 - f^2 (df)^2 = -(z^{\lambda^*} dz^\lambda - z^\lambda dz^{\lambda^*})^2/4, \]
then (4.1) reduces to the following
\[ (4.2) \quad \psi^*(ds_\alpha^2) = \frac{f^2}{(f+1)^2} ds_z^2 + \frac{1}{2f^2(f+1)^2} (z^{\lambda^*} dz^\lambda - z^\lambda dz^{\lambda^*})^2. \]
§5. The similarity and the inversion in $CP^m$. Let $\phi$ be a similarity at the origin in $E^{2m}$:

\[ \phi : \alpha^\lambda \rightarrow \alpha^\lambda = c \alpha^\lambda, \]

where $c$ is a positive constant.

A similarity $\tilde{\phi}$ of $CP^m$ at $O(\subseteq V_\Delta)$ will be defined by

\[ \tilde{\phi} = \psi^{-1} \circ \phi \circ \psi. \]

If we take account of (3.9), (3.10), (5.1) and

\[ z^\lambda = \frac{2c}{(1-c^2)(f+1+c^2)} z^\lambda, \]

\[ \alpha^2 = \frac{1}{f+1} z^\lambda, \]

\[ z^\lambda = \frac{2}{1-\alpha^c} \alpha^{c^*} \alpha^\lambda, \]

$\tilde{\phi}$ is given in terms of the local coordinate $\{z^\lambda\}$ in $V_\Delta$ as follows:

\[ (5.2) \]

Next we shall induce a transformation of $CP^m$ from an inversion in $E^{2m}$ by $\psi$. Consider an inversion $\phi$ in $E^{2m}$ with respect to a hypersphere of origin $O_\alpha$ and radius $r>0$, $\phi$ is given by an equation of the form

\[ \phi : \alpha^\lambda \rightarrow \alpha^\lambda = i \alpha^\lambda, \]

where $i$ is a positive-valued function. As $\phi$ satisfies

\[ (\alpha^\lambda \alpha^{c^*}) (\alpha^{c^*} \alpha^\mu) = r^4, \]

we get by virtue of (3.9)

\[ (5.3) \]

We shall define $\tilde{\psi}$ by $\tilde{\psi} = \psi^{-1} \circ \phi \circ \psi$ restricting the value of $r$ to sufficiently small and the domain of $\phi$ suitably. $\tilde{\phi}$ will be called an inversion of $CP^m$ at $O$. The expression of $\tilde{\phi}$ is found as follows. The equation (5.2) being still true for a non-constant $c$, we have

\[ z^\lambda = \frac{2c}{(1-c^2)(f+1+c^2)} z^\lambda. \]

If we substitute (5.3) into the last equation, it follows that

\[ z^\lambda = \frac{2r^2}{f-1-r^2(f+1)} z^\lambda. \]

Putting $c=\frac{1}{r^2}$, we get for $\tilde{\phi}$

\[ z^\lambda = \frac{2c}{(1-c^2)(f+1+c^2)} z^\lambda. \]
Comparing this equation with (5.2), we know that the similarity and the inversion of \( CP^m \) are given by an equation of the same form:

\[
(5.4) \quad z^\lambda = \frac{2c}{(1-c^2)f+1+c^2} z^\lambda, \quad c \neq 0.
\]

§ 6. \( F \)-transformation. Let \( C^m \) be the \( m \) dimensional complex number space with coordinate \( \{z^\lambda\} \). The Fubini metric is defined by

\[
(6.1) \quad ds_z^2 = 2g_{\mu\nu}dz^\mu dz^{\mu*} = \frac{1}{S^2(S\delta_{\lambda\mu} - 2kz^{\lambda *} z^{\mu})} dz^\lambda dz^{\mu*}
= -\frac{2}{S} ds^2 dz^\lambda - \frac{4k}{S^2} |z^{\lambda *} dz^\lambda|^2,
\]

where

\[
S = S(u) = 1 + 2ku, \quad u = z^\lambda z^{\lambda*},
\]

and \( k \) is a non-zero real constant.

Let \( F^m \) be the maximal domain of \( C^m \) in where \( S \) is positive, and we shall call \( \{F^m, ds_z^2\} \) a Fubini space which will be denoted by \( F^m \).

\( F^m \) is a Kähler space of constant holomorphic curvature.

Our purpose of this section is to generalize the discussions in § 4 and § 5 to \( F^m \).

Consider a transformation \( \phi \) of \( F^m \) such that

\[
(6.2) \quad \phi : z^\lambda \rightarrow w^\lambda = t(u)z^\lambda,
\]

where \( t \) is a real-valued differentiable function of \( u = z^\lambda z^{\lambda*} \).

It is known [8] that any geodesic through the origin \( O \) in \( F^m \) is given by

(i) for \( k > 0 \),

\[
z^\lambda = A^\lambda \tan(\sqrt{k} s),
\]

where \( A^\lambda \) are complex numbers satisfying \( 2kA^\lambda A^{\lambda*} = 1 \).

(ii) for \( k < 0 \),

\[
z^\lambda = A^\lambda \tanh(\sqrt{|k|} s),
\]

where \( A^\lambda \) are complex numbers satisfying \( 2kA^\lambda A^{\lambda*} = -1 \).

Thus \( \phi \) leaves invariant each geodesic through \( O \), and hence it is a geodesic transformation at \( O \) in the sense of [7].

First we shall get the relation between \( ds_z^2 \) and \( \phi^*(ds_w^2) \). If we put

\[
v = w^\lambda w^{\lambda*} = t^2 u,
\]

it holds that

\[
ds_w^2 = \left( \frac{2}{S(v)} dw^\lambda dw^{\lambda*} - \frac{4k}{S^2(v)} |w^{\lambda*} dw^\lambda|^2 \right).
\]
Hence, taking account of
\[ dw^2 = t'du + t' dz^2, \quad t' = dt/du, \]
\[ dw^2 = t'(t'u + t) du^2 + t^2 dz^2, \]
\[ w^2 dw^2 = t(t'u du + t' z^2 dz^2), \]
we have
\[ |w^2 dw^2|^2 = t^2 (t'u(t'u + t) du^2 + t^2 |z^2 dz^2|^2). \]

On the other hand, it follows from (6.1) that
\[ 2 dz^2 z^2 = S(u) du^2 + \frac{4k}{S(u)} |z^2 dz^2|^2. \]
Thus, if we substitute the last equation into (6.3), the following equation is obtained:
\[ \phi^*(ds^2_w) = \frac{t^2 S(u)}{S(v)} ds^2 + \frac{2}{S(u) S(v)} \mathbb{H}, \]
where
\[ \mathbb{H} = (1 + 2ku(t'u + t) du^2 + 2kt^2 (1 - t^2) |z^2 dz^2|^2 \]
\[ = (1 + 2ku)t'(t'u + t)(z^2 dz^2 - z^2 dz^2)^2 \]
\[ + 2 (1 + 2ku) t'(t'u + t) + kt^2 (1 - t^2) |z^2 dz^2|^2. \]

Now we shall call a transformation \( \phi \) of (6.2) an \( F \)-transformation, if \( t(u) \) satisfies
\[ t(u) = \frac{2c}{(1 - c^2) f + 1 + c^2}, \]
where \( c \) is a non-zero real constant and
\[ f = \sqrt{S(u)} = \sqrt{1 + 2ku}, \quad u = z^2 z^2. \]
This transformation is a generalization of the similarity and the inversion of CP^m.

For an \( F \)-transformation \( \phi \), the coefficient of \( |z^2 dz^2|^2 \) in (6.5) vanishes identically. In fact, it is proved as follows. If we put
\[ \rho = \rho(u) = (1 - c^2) f + 1 + c^2, \]
then
\[ t = \frac{2c}{\rho} \]
and
\[ 1 - t^2 = 2(1 - c^2) [(1 + c^2) f + (1 - c^2)(1 + ku)] / \rho^2. \]
hold good. As we have
\[ f' = k/f, \quad \rho' = (1 - \rho^2)f' = k(1 - \rho^2)/f, \]
it follows that
\[ t' = -2kc(1 - \rho^2)/\rho f, \quad (6.7) \]
\[ t'u + t = 2c((1 - \rho^2)(1 + ku) + (1 + \rho^2)f)/\rho f. \]
By (6.6) and (6.7) we can get
\[ 2(1 + 2k\mu)t'(t'u + t) + kt^2(1 - t^2) = 0, \]
which implies our assertion.

Thus we know that \( \Phi \) in (6.4) reduces to
\[ \Phi = (1 + 2k\mu)t'(t'u + t)(x^*dz^2 - z\lambda dz^*)^2 \]
for any \( F \)-transformation.

§ 7. The converse problem. Consider a transformation \( \phi \) in \( F^m - \{O\} \) given by (6.2), i.e.,
\[ \phi : z^2 \rightarrow w^2 = t(u)z^2. \]
We assume that \( \phi \) satisfies
\[ \phi^*(ds_u^2) = \frac{t^2S(u)}{S(v)}ds_z^2 + \sigma(x^*dz^2 - z\lambda dz^*)^2 \]
identically, where \( \sigma \) is a real-valued function.

The purpose of this section is to prove that the \( \phi \) is an \( F \)-transformation.

Under the assumption, the problem is reduced to solving the differential equation for \( t \):
\[ (7.1) \quad 2(1 + 2k\mu)t'(t'u + t) + kt^2(1 - t^2) = 0 \]
by virtue of (6.4) and (6.5).
If we put
\[ x = \sqrt{1 + 2k\mu}, \quad y = 1/t, \]
then (7.1) becomes the following equation:
\[ (7.2) \quad [(x^2 - 1)p - 2xy]p + y^2 - 1 = 0, \quad p = dy/dx. \]
Differentiating (7.2) with respect to \( x \), we have
\[ [(x^2 - 1)p - xy] \frac{dp}{dx} = 0 \]
and hence
\[ (i) \quad \frac{dp}{dx} = 0 \quad \text{or} \quad (ii) \quad \frac{p}{y} = \frac{x}{x^2 - 1}. \]
Case (i). It follows that \( y = Cx + D \), where \( C \) and \( D \) are constant. Substituting
this form of $y$ into (7.2) we get $D^2 = C^2 + 1$ and
\[ y = Cx \pm \sqrt{C^2 + 1}. \]
Therefore if we put
\[ c = -C \pm \sqrt{C^2 + 1}, \]
we can get
\[ t = \frac{2c}{(1 - c^2)x + 1 + c^2} \]
which shows that $\phi$ under consideration is an $F$-transformation.

Case (ii). By integration, we have
\[ (7.3) \quad \log |y| = \frac{1}{2} \log |x^2 - 1| + C. \]
If $k > 0$, (7.3) gives $y = C\sqrt{x^2 - 1}$ which and (7.2) lead us to a contradiction $C^2 + 1 = 0$. If $k < 0$, we have $y = C\sqrt{1 - x^2}$, and by (7.2) $C = \pm 1$ follows. Therefore we have
\[ t = \pm 1/\sqrt{1 - x^2} = \pm 1/\sqrt{-2ku}. \]
Consequently it follows that
\[ w^2 z^2 = t^2 z^2 x^2 = -1/2k \]
which is contradictory to the diffeomorphism of $\phi$.

REMARK. In $E^{2m}$, let $\phi: z^\lambda \to w^\lambda = \ell(u)z^\lambda$ be a (local or global) diffeomorphism. Then it is easy to see that $\phi$ satisfies
\[ \phi^*(dz_w^2) = \rho d\bar{z}^2 + \sigma(z^\lambda dx^\lambda - x^\lambda dz^\lambda)^2 \]
if and only if $\phi$ is a similarity or an inversion with or without composition of the symmetry at $O$, where $\rho$ and $\sigma$ are real-valued functions of $\{z^\lambda\}$. In this case, $\sigma$ actually vanishes.

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