*Journal of the Korean Nuclear Society*
*Volume 5, Number 2, June, 1973*

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# Critical Mass Minimization of a Cylindrical Geometry Reactor by Two Group Diffusion Equation

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(Received March 19, 1973)

**Abstract**

L. S. Pontryagin's maximum principle is applied to the minimum critical mass problem without any restriction on the ranges of uranium enrichment. For the analysis, two group diffusion equation is adopted for a cylindrical reactor neglecting the vertical axis consideration. The result shows that the three-zoned reactor turns out to be most optimal: the inner and outer zones with the minimum enrichment; whereas the middle zone with the maximum enrichment. With the given three-zoned reactor, critical condition is derived, which leads to the calculation of the determinant. By finding the roots of the determinant the numerical calculation of the minimum critical mass is carried out for the case of Kori reactor geometry changing the minimum or the maximum enrichment. It is found from many computed values that the least possible critical mass turns out to be the case of 1.2% maximum enrichment for the middle zone and 0.65% minimum enrichment for the inner and out zones.

**요 약**

L. S. Pontryagin의 Maximum Principle과 수직방향을 고려하지 않은 2군 확산 방정식을 우라늄농축도 범위에 제한없이 원통형원자로의 최소 임계질량문제에 적용하였다. 핵연료 장전방법에 관한한 최적 원자로는 내심부와 외심부가 최소의 농축도를 갖고 중간영역은 최대의 농축도를 갖는 3-영역식 원자로인 것으로 밝혀졌다.

상기 3-영역식 원자로를 모델로 하여 임계조건을 유도하였으며, 또한 고리원자로를 예로하여 농축도를 여러가지로 변환시키면서 임계조건의 해를 구하는 수치해석을 수행하였다. 그 결과 여러가지 임계조건중 최소의 임계질량을 갖는 경우는 중간영역에서의 최대 농축도가 1.2%이고 내심부와 외심부에서의 농축도가 0.65%일때라는 것이 판명되었다.

## 1. Introduction

In recent years a great deal effort has been employed for the development of optimization problems which are encountered in the field of nuclear reactor theory. Attempt was made in the past for finding the optimal conditions in nuclear engineering, *i.e.*, power maximiza-

tion, critical mass minimization, burnup maximization, refueling optimization, etc., by means of mathematical approach.[1-6]

For instance, Suzuki[2] made an effort to solve the problem of fuel burnup maximization by topological mapping theory and classical calculus of variation, while Sauar[3] made use of linear program for the solution of in-core fuel management optimization. On the other hand, a good many number of scientists and research engineers have adopted the maximum principle by Pontryagin in tackling their optimization topics since Kochurov[4] introduced that methodology in 1965 for the calculation of minimum critical mass of a nuclear reactor.

Kochurov has verified, using a two-group diffusion equation in slab and spherical geometry reactor, that the optimal critical configuration for a thermal reactor is made up of a three-zoned system: namely, the inner zone with the maximum permissible uranium concentration; the middle zone with such a uranium distribution which corresponds to the constant thermal neutron density; and the reflector. Using a two group diffusion model in slab geometry, Zaritskaya and Rudik[5] applied in 1967 Pontryagin's maximum principle to seek for the minimum criticial dimension of a thermal reactor having a given total power, with constraints on the maximum power density as well as on the maximum fuel enrichment. The clarification of relationship between the minimum critical size and the maximum power is attributed to their work.

Goldschmidt[1] extended, in his thesis, Kochurov's work to minimum critical mass problem of a slab reactor having a given power which is subject to constraints on the maximum power density and on the maximum fuel enrichment. His former paper[6] also shows the distribution of fuel enrichment which minimizes the critical mass of a fast reactor.

It is attempted in this paper to find the minimum critical mass arrangement in a cylindrical reactor making use of Pontryagin's maximum principle[7] which has enjoyed widespread uses in the areas of control problem. The basis of the maximum principle is placed on the generalization of the Lagrange problem in the calculus of variations. The philosophy of this principle is to find the very condition that optimizes the functional we consider. Unfortunately, however, this princiciple only gives necessary conditions, and fails to provide us with an answer as to how many zones of reactor core or what kind of enrichment there should be in the core: which is to say that there is no algorithm for them.

To help understand the maximum principle to a certain extent, an outline of this theory is formulated in a section of this paper. At the same time, effort is made to apply this principle to the two group diffusion equation so as to find the optimal distribution of fuel arrangement that minimizes the critical mass of a cylindrical reactor.

Apart from other papers previously published, it may be regarded as the unique achievement of this paper that no restriction on the uranium enrichment is adopted in performing this work.

The optimal condition for the fuel distribution draws our keen attention in this paper, and, at the same time, the types of fuel arrangement which satisfy the maximum principle are repeatedly tested so as to verify at what condition they fall under the optimal case. Finally the critical condition is imposed on an optimal reactor, and on the case of Kori reactor geometry as well.

## 2. The Maximum Principle

In accordance with L. S. Pontryagin, let us proceed to the formulation of the theorem

which yields the solution of the fundamental problem. Hence a system which can be expressed by differential equations is presumed to be as follows:

$$\frac{dx^i}{dt}=f^i(x^1, x^2, \cdots x^n, u^1, u^2, \cdots u^r)=f^i(\vec{x}, u),$$
$$i=1, 2, 3, \cdots n, \tag{1}$$

or in vector form,

$$\frac{d\vec{x}}{dt}=\vec{f}(\vec{x}, u) \tag{2}$$

where $\vec{f}(\vec{x}, u)$ is the vector with coordinates

$$f^1(\vec{x}, u), f^2(\vec{x}. u), \cdots\cdots f^n(\vec{x}, u).$$

It is supposed that an additional function $f^0(x^1, x^2, \cdots x^n, u)=f^0(\vec{x}, u)$ is given, and that the functions, $f^i$, $i=0, 1, 2\cdots n$, are assumed to be continuous in the variables $x^1, x^2, \cdots x^n, u$, and are continuously differentiable with respect to $x^1, x^2, \cdots x^n$. In this case, the fundamental problem (finding the optimal controls) can be formulated as follows:

"Among all the admissible controls, $u=u(t)$, effort is made to find one for which the functional

$$J=\int_{t_0}^{t_1} f^0\{\vec{x}(t), u(t)\}dt \tag{3}$$

takes the least possible value."

In order to cope with the optimality condition, it will be convenient to reformulate the present problem, in such a way that a new coordinate, $x^0$, which varies according to the following law, is defined:

$$\frac{dx^0}{dt}=f^0(\vec{x}, u). \tag{4}$$

Then the fundamental system of Equation (1) is reduced to

$$\frac{dx^i}{dt}=f^i(\vec{x}, u), \quad i=0, 1, 2, \cdots n. \tag{5}$$

In addition another system of equations is defined in the auxiliary variables, $\Psi_0, \Psi_1, \cdots \Psi_n$,

$$\frac{d\Psi_i}{dt}=-\sum_{\alpha=0}^{n}\frac{\partial f^\alpha(\vec{x}, u)}{\partial x^i}\Psi_\alpha,$$
$$i=0, 1, 2, \cdots\cdots n, \tag{6}$$

Since this system is linear and homogeneous, it admits, for any initial conditions, the unique solution

$$\vec{\Psi}=(\Psi_0, \Psi_1, \Psi_2, \cdots\cdots \Psi_n)$$

for the $\Psi_i$ which is defined on the entire integral $t_0\leq t\leq t_1$ on which $u(t)$ and $x^i$, $i=0, 1, 2, \cdots n$, are defined.

The above two systems, namely, (5) and (6), are now combined into one entry. In order to do so, the following function $H$ of the variables, $x^1, x^2, \cdots x^n, \Psi_0, \Psi_1, \cdots \Psi_n, u^1, u^2, \cdots u^r$, are considered.

$$H(\vec{\Psi}, \vec{x}, u)=\{\vec{\Psi}, \vec{f}(\vec{x}, u)\}=\sum_{\alpha=0}^{n}\Psi_\alpha f^\alpha(\vec{x}, u). \tag{7}$$

It would be easily verified that the above systems, (5) and (6), could be rewritten with the aid of this function in the form of the following Hamiltonian system:

$$\frac{dx^i}{dt}=\frac{\partial H}{\partial \Psi_i}, \quad i=0, 1, 2, \cdots\cdots n, \tag{8}$$

$$\frac{d\Psi_i}{dt}=-\frac{\partial H}{\partial x^i}, \quad i=0, 1, 2, \cdots\cdots n. \tag{9}$$

Then, so far as the optimal control problem is concerned, the maximum principle of Pontryagin can be stated as follows:

"In order that $u(t)$ and $x^i(t)$, $i=0, 1, \cdots n$, be optimal, it is necessary that there exists non-zero continuous vector function, $\vec{\Psi}(t)=\{\Psi_0(t), \Psi_1(t), \cdots \Psi_n(t)\}$ corresponding to $u(t)$ and $x^i(t)$, $i=0, 1, 2, \cdots n$, such that:

1. For every $t$, $t_0\leq t\leq t_1$, the function $H\{\vec{\Psi}(t), \vec{x}(t), u\}$ of the variable $u$ attains its maximum at the point $u=u(t)$:
$$H\{\vec{\Psi}(t), \vec{x}(t)\}=M\{\vec{\Psi}(t), \vec{x}(t)\}; \tag{10}$$
2. At the terminal point $t_1$ the relations
$$\Psi_0(t_1)\leq 0, \tag{11a}$$
$$M\{\Psi(t_1), \vec{x}(t_1)\}=0, \tag{11b}$$
are satisfied. If $\frac{d\Psi_0}{dt}=0$ for $t_0\leq t\leq t_1$,
$$\Psi_0=\text{constant}\leq 0."$$

This theorem is extremely useful in the case of autonomous system (*i.e.*, the Hamiltonian $H$ does not depend explicitly on the variable $t$), but it can be extended to the nonautonomous system. In this

paper, however, only the results both of autonomous and nonautonomous systems are being utilized.

### 3. Two Group Diffusion Equation

Two group approximation[8] is to suppose that the neutrons may be divided into two energy groups, namely, thermal (slow) and fast. In the steady state condition, two group diffusion equation which describes the fast and thermal neutron fluxes is given by

$$\nabla(D_1\nabla\phi_1)-\Sigma_1\phi_1+\frac{k_\infty}{p}\Sigma_2\phi_2=0, \quad (12a)$$

$$\nabla(D_2\nabla\phi_2)-\Sigma_2\phi_2+p\Sigma_1\phi_1=0, \quad (12b)$$

where $D_1$ : diffusion coefficient for fast neutrons

$D_2$ : diffusion coefficient for thermal neutrons

$\Sigma_1$ : macroscopic scattering cross-section of fast neutrons for moderator

$\Sigma_2$ : macroscopic absorption cross-section of thermal neutrons for U-235

$k_\infty$ : infinite multiplication factor

$p$ : resonance escape probability.

In the above treatment, the fast neutrons are represented by the subscript 1, whereas the thermal neutrons by the subscript 2.

As is self-explanatory, the Equation (12) is a coupled balance equation for fast and thermal neutrons. In a thermal reactor, the fast group neutrons are resulted from fissions induced by $\Sigma_2\phi_2$ thermal neutrons absorbed in fuel. Thus, at the point $\vec{r}$, $\eta\varepsilon\Sigma_{2f}\phi_2(\vec{r})$ fast neutrons are produced per cm$^3$-sec. At the same time $\Sigma_1\phi_1$ neutrons are lost from the fast group due to slowing down process. As the result of slowing down out of the fast group, it is taken for granted that the slowing down density $p\Sigma_1\phi_1$ in the fast group turns out to be the source term in the thermal diffusion equation. For the simplicity, the following assumption is imposed on the two group constants, which is supposed to be reasonable and compatible in reactor theory.

Assumption:

1. $D_1$ and $D_2$ are constants (these are, in general, slow-varying functions for a given moderator).
2. $\Sigma_1$ is constant.
3. $\Sigma_2=(\sigma_a{}^{235}N^{235}+N^{others}\sigma_a{}^{others})=(\gamma u'+\delta)=u$, where $\sigma_a$ is microscopic absorption cross-section for $U^{235}$,

$$\gamma=\sigma_a{}^{235}(N^{235}+N^{238}),$$

$$\delta=N^{others}\sigma_a{}^{others},$$

and $u'$ is the uranium enrichment, that is, $\frac{N^{235}}{N^{235}+N^{238}}$. Equation (12) is now reduced to Equations (13a) and (13b), respectively.

$$\nabla^2\phi_1-\frac{1}{\tau_1}\phi_1=-\alpha u\phi_2, \quad (13a)$$

$$\nabla^2\phi_2-\beta u\phi_2=-\frac{1}{\tau_2}\phi_1, \quad (13b)$$

where

$$\frac{\Sigma_1}{D_1}=\frac{1}{\tau_1},$$

$$\frac{k_\infty}{pD_1}=\alpha,$$

$$\frac{p\Sigma_1}{D_2}=\frac{1}{\tau_2},$$

$$\frac{1}{D_2}=\beta.$$

Applying Equation (13) to a cylindrical reactor and using Pontryagin's maximum principle, work herein is directed toward seeking the optimum control variable which minimizes the total uranium mass to be loaded in the core. When $u$ is only dependent on radius $r$, independent of vertical axis $z$ in a cylindrical reactor, the total uranium quantity to be loaded is linearly related to

$$J=2\pi\int_0^R u(r)r\,dr, \quad (14)$$

where $R$ is the radius of the cylindrical reactor.

Let us consider a two group solution of the

radial problem for finding the minimum critical mass, *i.e.*, the minimum of the following functional $J$:

$$J=2\pi\int_0^R u(r)rdr.$$

Hence the notations such as the following are introduced:

$$\begin{aligned} x^0&=\int_0^r u(r)rdr;\\ x^1&=\phi_1,\\ x^2&=r\frac{d\phi_1}{dr},\\ x^3&=\phi_2,\\ x^4&=r\frac{d\phi_2}{dr}. \end{aligned} \tag{15}$$

Then the two group diffusion equation may be written in the form of:

$$\frac{dx^0}{dr}=2\pi ur, \tag{16a}$$

$$\frac{dx^1}{dr}=\frac{1}{r}x^2, \tag{16b}$$

$$\frac{dx^2}{dr}=\frac{1}{\tau_1}rx^1-\alpha urx^3 \tag{16c}$$

$$\frac{dx^3}{dr}=\frac{1}{r}x^4, \tag{16d}$$

$$\frac{dx^4}{dr}=\beta urx^3-\frac{1}{\tau_2}rx^1. \tag{16e}$$

The Hamiltonian of the system (16) has the form of:

$$\begin{aligned} H=(\vec{\Psi},\vec{f})&=u(2\pi r\Psi_0-\alpha r\Psi_2x^3+\beta r\Psi_4x^3)\\ &+\Big(\frac{1}{r}\Psi_1x^2+\frac{1}{\tau_1}r\Psi_2x^1+\frac{1}{r}\Psi_3x^4\\ &-\frac{1}{\tau_2}r\Psi_5x^1\Big)=u\varphi+\chi, \end{aligned} \tag{17}$$

where the auxiliary functions $\Psi_i$ $(i=0, 1, 2, 3, 4)$ satisfy the equations

$$\frac{d\Psi_0}{dr}=-\frac{\partial H}{\partial x^0}=0, \tag{18a}$$

$$\frac{d\Psi_1}{dr}=-\frac{\partial H}{\partial x^1}=-\frac{1}{\tau_1}r\Psi_2+\frac{1}{\tau_2}r\Psi_4, \tag{18b}$$

$$\frac{d\Psi_2}{dr}=-\frac{\partial H}{\partial x^2}=-\frac{1}{r}\Psi_1, \tag{18c}$$

$$\frac{d\Psi_3}{dr}=-\frac{\partial H}{\partial x^3}=\alpha ur\Psi_2-\beta ur\Psi_4, \tag{18d}$$

$$\frac{d\Psi_4}{dr}=-\frac{\partial H}{\partial x^4}=-\frac{1}{r}\Psi_3. \tag{18e}$$

## 4. Admissible Types of Control

When $u(r)$ is considered a variable over the volume of the reactor within the limits:

$$u_{min}\leq u(r)\leq u_{max}.$$

Pontryagin's maximum principle requires that, in order to find optimal distribution $u(r)$, a vector function $\vec{\Psi}_i$ $(i=0, 1, 2, 3, 4)$ must be sought, which is continuous and not identically equal to zero, such that the Hamiltonian $H$ as a function of the independent variable $u(r)$, reaches a maximum at every point in the region $0\leq r\leq R$.

In consequence of the linear dependence (17) of the Hamiltonian on $u$, it is clear that the following types of control are admissible:

$$u(r)=u_{min} \quad \text{for } \varphi=(2\pi r\Psi_0-\alpha r\Psi_2x^3+\beta r\Psi_4x^3)<0, \tag{19a}$$

$$u(r)=u_{max} \quad \text{for } \varphi=(2\pi r\Psi_0-\alpha r\Psi_2x^3+\beta r\Psi_4x^3)>0 \tag{19b}$$

because of the fact that, in this case, the Hamiltonian, as a function of the argument $u$, reaches a maximum.

Together with the differential equation for $x^i$, boundary conditions must be given for the unique solution to be determined. Boundary conditions based on the physical argument in reactor theory [8-12] are:

1. $x^1(R)=x^3(R)=0,$ (20a)
2. $x^2(0)=x^4(0)=0,$ (20b)

where $R$ is the exptrapolated radius. Now, how is it possible to impose boundary condition on $\Psi_i$?

With regard to this matter, Pontryagin does suggest that $\Psi_i$ must satisfy the transversality conditions[7] at the end points, $r=0, r=R$, that is,

$$\begin{aligned} &\Psi_1(0)dx^1|_{r=0}+\Psi_2(0)dx^2|_{r=0}+\Psi_3(0)dx^3|_{r=0}\\ &\qquad+\Psi_4(0)dx^4|_{r=0}=0 \end{aligned} \tag{21a}$$

and

$$\begin{aligned} &\Psi_1(R)dx^1|_{r=R}+\Psi_2(R)dx^2|_{r=R}+\Psi_3(R)dx^3|_{r=R}\\ &\qquad+\Psi_4(R)dx^4|_{r=R}=0, \end{aligned} \tag{21b}$$

where $r$ is parameter,

so $dx^1(R)=dx^3(R)=dx^2(0)=dx^4(0)=0$

($\because x^1(R)=x^3(R)=x^2(0)=x^4(0)=0$).

From the transversality conditions,

1. $\Psi_1(0)=\Psi_3(0)=0$, (22a)

2. $\Psi_2(R)=\Psi_4(R)=0$. (22b)

We infer from the maximum principle that $\Psi_0=\text{const.}\leq 0$ (it is, however, seen that $\Psi_0$ cannot be zero, owing to the fact that, if $\Psi_0=0$, the quantity $u(r)r$ in Hamiltonian that we are going to minimize vanishes).

Since the system of equations for $\Psi_i$ is homogeneous, $\Psi_0$ can be normalized so that $2\pi\Psi_0=-1$. Then the function $\varphi$ becomes:

$$\varphi(r)=-r(1+\alpha\Psi_2 x^3-\beta\Psi_4 x^3). \quad (23)$$

From the boundary conditions (20) and the transversality conditions (21), it is easily verified that

$$\varphi(0)=0 \quad (24a)$$

and

$$\varphi(R)=-R<0. \quad (24b)$$

The fuction $\varphi(r)$ passes through zero at $r=0$ and is negative at $r=R$. In order to determine the regions of

$$\varphi<0(u=u_{min}) \text{ and } \varphi>0\ (u_{max}),$$

this problem is investigated in some detail. Equation (24) means that, considering an optimal region arrangement with the boundaries at $r=0$ and $r=R$, the function $\varphi(r)$ has many optimal possibilities as shown in Figure 1.

Fig. 1 shows various possible types of $\varphi(r)$ and their corresponding uranium enrichments which satisfy the condition (19). Hence the maximum principle implies that each fuel arrangement satisfies the necessary condition for the reactor to have the minimum critical mass. Unfortunately, however, this principle does not clarify as to which is the most optimal one.

Therefore, all the possible arrangements should be analysed with the aid of boundary

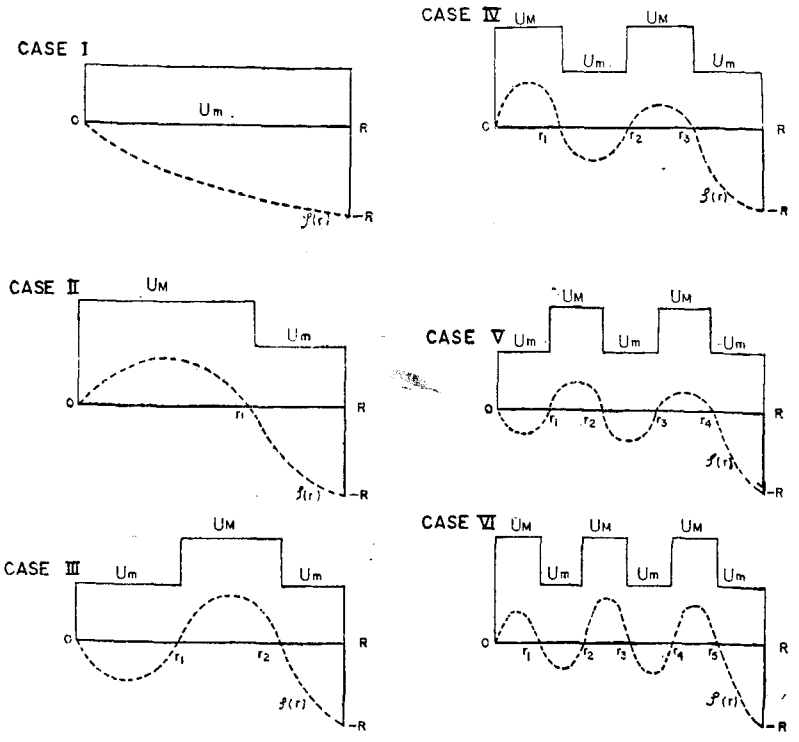


**Fig. 1. Possible optimal arrangement**

conditions or physical meaning. Taking only boundary conditions into account, each fuel arrangement must be given thorough check so as to find out whether such an arrangement satisfies Pontryagin's maximum principle or not. In other words, each case of fuel arrangement is tested in an attempt to check whether or not there exists a unique non-zero continuous vector function $\Psi_i$ $(i=0,1,2,3,4)$ corresponding to uranium enrichment $u(r)$ and $x^i(r)(i=0,1,2,3,4)$. In order to obtain the solutions $x^i$ and $\Psi_i$ which are made into being in the zones of each fuel arrangement, let us go back to the original equations (13) and (18), which read:

$$\nabla^2 x^1-\frac{1}{\tau_1}x^1=-\alpha u x^3, \tag{25a}$$

$$\nabla^2 x^3-\beta u x^3=-\frac{1}{\tau_2}x^1, \tag{25b}$$

$$\frac{d\Psi_0}{dr}=0,\ (\Psi_0=\text{constant}<0) \tag{26a}$$

$$\frac{d\Psi_1}{dr}=-\frac{1}{\tau_1}r\Psi_2+\frac{1}{\tau_2}r\Psi_4, \tag{26b}$$

$$\frac{d\Psi_2}{dr}=-\frac{1}{r}\Psi_1, \tag{26c}$$

$$\frac{d\Psi_3}{dr}=\alpha u r\Psi_2-\beta u r\Psi_4 \tag{26d}$$

$$\frac{d\Psi_4}{dr}=-\frac{1}{r}\Psi_3. \tag{26e}$$

Eliminating $\Psi_1$ and $\Psi_3$ from Equation (26), the relations between $\Psi_2$ and $\Psi_4$ can be obtained, which are quite analogous to those between $x^1$ and $x^3$,

Substituting the following relations into (26b) and (26d),

$$\Psi_1=-r\frac{d\Psi_2}{dr},$$

and $\Psi_3=-r\frac{d\Psi_4}{dr}$ [from (26c) and (26e)],

we obtain the below-mentioned equation.

$$\nabla^2\Psi_2-\frac{1}{\tau_1}\Psi_2=-\frac{1}{\tau_2}\Psi_4, \tag{27a}$$

$$\nabla^2\Psi_4-\beta u\Psi_4=-\alpha u\Psi_2, \tag{27b}$$

The solutions of Equations (25) and (27) are easily obtainable by the standard method[8, 9, 10].

As shown in Equation (25), the thermal flux $x^3$ is seen in the equation representing the fast flux $x^1$ and, conversely, $x^1$ is seen in the equation for the thermal flux $x^3$. These equations can be decoupled by preferentially solving the Equation (25b) for $x^1$:

$$x^1=-\tau_2(\nabla^2 x^3-\beta u x^3). \tag{28a}$$

By the substitution of this expression into (25a), a fourth-order equation is obtained for $x^3$ alone as follows:

$$-\tau_2\nabla^2(\nabla^2 x^3-\beta u x^3)+\frac{\tau_2}{\tau^1}(\nabla^2 x^3-\beta u x^3)=-\alpha u x^3. \tag{28b}$$

Equation (28b) is rewritten:

$$\nabla^2\nabla^2 x^3-\left(\beta u+\frac{1}{\tau_1}\right)\nabla^2 x^3+u\left(\frac{\alpha}{\tau_2}-\frac{\beta}{\tau_1}\right)=(\nabla^2+\mu^2)(\nabla^2-\lambda^2)x^3=0, \tag{29}$$

where

$$\mu^2=\frac{1}{2}\left[-\left(\beta u+\frac{1}{\tau_1}\right)+\sqrt{\left(\beta u+\frac{1}{\tau_1}\right)^2+4u\left(\frac{\alpha}{\tau_2}-\frac{\beta}{\tau_1}\right)}\right]$$

and

$$\lambda^2=\frac{1}{2}\left[\beta u+\frac{1}{\tau_1}+\sqrt{\left(\beta u+\frac{1}{\tau_1}\right)^2+4u\left(\frac{\alpha}{\tau_2}-\frac{\beta}{\tau_1}\right)}\right].$$

Let the solutions for the themal flux $x^3$ coresponding to the two values, $\mu^2$ and $\lambda^2$ be $X$ and $Y$, respectively, then these equations may be written as the following expression:

$$\nabla^2 X+\mu^2 X=0, \tag{30a}$$

and

$$\nabla^2 Y-\lambda^2 Y=0. \tag{30b}$$

Each solution of $X$ and $Y$ consists of two independent functions since Equations (30a) and (30b) are both second-ordered. The linear combination then gives for the thermal flux $x^3$:

$$x^3=AY+CY. \tag{31a}$$

Applying the same method to the fast flux $x^1$, $x^1$ can now be found in the same form as follows:

$$x^1=A'X+C'Y. \tag{31b}$$

These solutions for $x^1$ and $x^3$ appear to be involved with four arbitary constants ($A, C, A'$

and $C'$).

It is, however, shown that only two out of four constants are independent. When the solution for $x^3$ is substituted into the Equation (25b), it is readily seen that:

$$\frac{A'}{A}=S'=\text{constant},$$

$$\frac{C'}{C}=S''=\text{constant}.$$

The quantities $S'$ and $S''$ are called the coupling coefficients[8, 9, 10, 12] designated with two group constants for the fast and thermal fluxes, respectively, and these are found in the next section of this paper.

In the cylindrical geometry, the solutions $X$ and $Y$ which satisfy the Equations (30a) and (30b) are zero-order Bessel function and zero-order modified Bessel function[13], respectively:

$$(\nabla^2+\mu^2)X=\frac{d^2X}{dr^2}+\frac{1}{r}\frac{dX}{dr}+\mu X^2=0$$

: zero-order Bessel equation (32a)

$$(\nabla^2-\lambda^2)Y=\frac{d^2Y}{dr^2}+\frac{1}{r}\frac{dY}{dr}-\lambda^2Y=0$$

: zero-order modified Bessel equation. (32b)

The solution $X$ is represented by $\{J_o(\mu r), Y_o(\mu r)\}$ and $Y$ by $\{I_o(\lambda r), K_o(\lambda r)\}$, where each bracket represents a linear combination of the two functions inside of it.

Therefore, the fast flux $x^1$ and the thermal flux $x^3$ are linear combination of the four functions $\{J_o(\mu r), Y_o(\mu r), I_o(\lambda r),$ and $K_o(\lambda r)\}$:

$$x^1=aS^1J_0(\mu r)+bS^2Y_0(\mu r)+cS^3I_0(\lambda r)+dS^4K_0(\lambda r), \quad (33a)$$

and

$$x^3=aJ_0(\mu r)+bY_0(\mu r)+cI_0(\lambda r)+dK_0(\lambda r), \quad (33b)$$

where $S^1, S^2, S^3$ and $S^4$ are coupling coefficients as explained above.

Now turning to the Equations (27a) and (27b), let us solve the equations for $\Psi_2$ and $\Psi_4$. Since $\varphi$ does not include $\Psi_1$ and $\Psi_3$, it is unnecessary to get information for $\Psi_1$ and $\Psi_3$ in this subject.

From Equation (27a),

$$\Psi_4=-\tau_2\nabla^2\Psi_2+\frac{\tau_2}{\tau_1}\Psi_2. \quad (34)$$

As in the case of $x^1$ and $x^3$, the substitution of the equation into (27b) makes it possible to obtain a fourth-order equation for $\Psi_2$:

$$\nabla^2\left[-\tau_2\left(\nabla^2\Psi_2-\frac{1}{\tau_1}\Psi_2\right)+\beta u\tau_2\left(\nabla^2\Psi_1-\frac{1}{\tau_1}\Psi_2\right)\right]=-\alpha u\Psi_2. \quad (35)$$

Equation (35) is rearranged:

$$\nabla^2\nabla^2\Psi_2-\left(\frac{1}{\tau_1}+\beta u\right)\nabla^2\Psi_2-\left(\frac{1}{\tau_2}\alpha u-\frac{1}{\tau_1}\beta u\right)\Psi_2=(\nabla_2+\mu^2)(\nabla^2-\lambda^2)\Psi_2=0. \quad (36)$$

The solution of auxiliary variables $\Psi_2$ and the thermal flux $x^3$ has the same form as well, which is somewhat remarkable fact to notice. Therefore, the solution of the auxiliary variables $\Psi_2$ and $\Psi_4$ are written in the following form:

$$\Psi_2=AJ_0(\mu r)+BY_0(\mu r)+CI_0(\lambda r)+DK_0(\lambda r) \quad (37a)$$

$$\Psi_4=AT^1J_0(\mu r)+BT^2Y_0(\mu r)+CT^3I_0(\lambda r)+DT^4K_0(\lambda r), \quad (37b)$$

where $T^1, T^2, T^3$ and $T^4$ are coupling constants.

For an arbitrary control $u(r)$ to be optimal, in accordance with the maximum principle, there must exist a non-zero unique continuous vector function $\Psi_i(i=0, 1, 2, 3, 4)$ corresponding to $u(r)$ and $x^i$ $(i=0, 1, 2, 3, 4)$.

## 5. Determination of Optimal Fuel Arrangement

Now let us investigate further so as to clarify what kind of fuel arrangement would satisfy the Pontryagin's maximum principle. Each case in Fig. 1 is analysed as follows:

Case 1) : One-zoned arrangement

$$x^1=A_1S_1^1I_o(\mu_m r)+B_1S_1^2Y_o(\mu_m r)+C_1S_1^3I_o(\lambda_m r)+D_1S_1^4K_0(\lambda_m r) \quad (38a)$$

$$x^3=A_1J_0(\mu_m r)+B_1Y_0(\mu_m r)+C_1I_0(\lambda_m r)+D_1K_0(\lambda_m r), \quad (38b)$$

$$\Psi_2=a_1T_1^1I_o(\mu_m r)+b_1T_1^2Y_0(\mu_m r)+c_1T_1^3I_0(\lambda_m r)+d_1T_1^4K_o(\lambda_m r), \quad (38c)$$

$$\Psi_4=a_1J_0(\mu_m r)+b_1Y_0(\mu_m r)$$

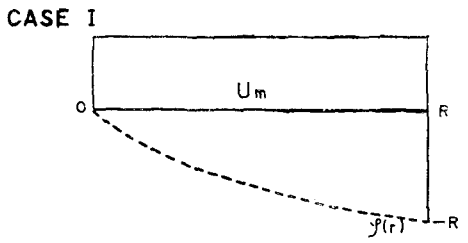


**Fig. 2. (a) one-zoned arrangement**

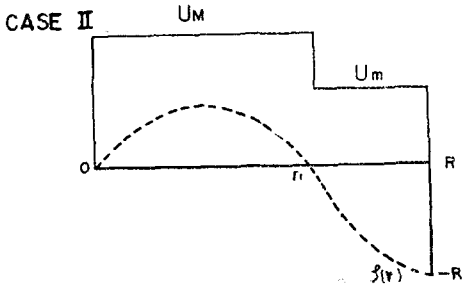


**Fig. 2. (b) Two-zoned arrangement**

$$+c_1I_0(\lambda_m r)+d_1K_0(\lambda_m r), \tag{38d}$$

where

$$\mu_m=\left[\frac{1}{2}\left\{-\left(\beta u_m+\frac{1}{\tau_1}\right)+\sqrt{\left(\beta u_m+\frac{1}{\tau_1}\right)^2+4u_m\left(\frac{\alpha}{\tau_2}-\frac{\beta}{\tau_1}\right)}\right\}\right]^{1/2}, \tag{38e}$$

$$\lambda_m=\left[\frac{1}{2}\left\{\beta u_m+\frac{1}{\tau_1}+\sqrt{\left(\beta u_m+\frac{1}{\tau_1}\right)^2+4u_m\left(\frac{\alpha}{\tau_2}-\frac{\beta}{\tau_1}\right)}\right\}\right]^{1/2}. \tag{38f}$$

Boundary conditions:

$$x^1(R)=x^3(R)=0, \tag{39a}$$

$$\left.\frac{dx^1}{dr}\right|_{r=0}=\left.\frac{dx^3}{dr}\right|_{r=0}=0, \tag{39b}$$

$$\Psi_2(R)=\Psi_4(R)=0. \tag{39c}$$

Hence the four boundary conditions for $x^1$ and $x^3$ completely determine the unknown constants $A_1, B_1, C_1$, and $D_1$. The unknown constants, $a_1$, $b_1$, $c_1$ and $d_1$, can be determined but not uniquely by the two boundary conditions for $\Psi_2$ and $\Psi_4$. This satisfies the necessary condition for optimality. It is, however, clear from the viewpoint of the reactor analysis that the uniform reactor like this one cannot be optimal. Consequently, the one-zoned arrangement is not attempted for analysis.

Case 2) : Two-zoned arrangement

$(0\leq r\leq r_1)$

$$x^1=A_1S_1^1I_0(\mu_M r)+B_1S_1^2Y_0(\mu_M r)+C_1S_1^3I_0(\lambda_M r)+D_1S_1^4K_0(\lambda_M r), \tag{40a}$$

$$x^3=A_1I_0(\mu_M r)+B_1Y_0(\mu_M r)+C_1I_0(\lambda_M r)+D_1K_0(\lambda_M r), \tag{40b}$$

$(r_1\leq r\leq R)$

$$x^1=A_2S_2^1I_0(\mu_m r)+B_2S_2^2Y_0(\mu_m r)+C_2S_2^3I_0(\lambda_m r)+D_2S_2^4K_0(\lambda_m r) \tag{40c}$$

$$x^3=A_2I_0(\mu_m r)+B_2Y_0(\mu_m r)+C_2I_0(\lambda_m r)+D_2K_0(\lambda_m r), \tag{40d}$$

and

$(0\leq r\leq r_1)$

$$\Psi_2=a_1T_1^1I_0(\mu_M r)+b_1T_1^2Y_0(\mu_M r)+C_1T_1^3I_0(\lambda_M r)+d_1T_1^4K_0(\lambda_M r), \tag{41a}$$

$$\Psi_4=a_1J_0(\mu_M r)+b_1Y_0(\mu_M r)+c_1I_0(\lambda_M r)+d_1K_0(\lambda_M r), \tag{41b}$$

$(r_1\leq r\leq R)$

$$\Psi_2=a_2T_2^1J_0(\mu_m r)+b_2T_2^2Y_0(\mu_m r)+c_2T_2^3I_0(\lambda_m r),+d_2T_2^4K_0(\lambda_m r), \tag{41c}$$

$$\Psi_4=a_1J_0(\mu_m r)+b_2Y_0(\mu_m r)+c_2I_0(\lambda_m r)+d_2K_0(\lambda_m r), \tag{41d}$$

where

$$\mu_M=\left[\frac{1}{2}\left\{-\left(\beta u_M+\frac{1}{\tau_1}\right)+\sqrt{\left(\beta u_M+\frac{1}{\tau_1}\right)^2+4u_M\left(\frac{\alpha}{\tau_2}-\frac{\beta}{\tau_1}\right)}\right\}\right]^{1/2} \tag{41e}$$

$$\lambda_M=\left[\frac{1}{2}\left\{\beta u_M+\frac{1}{\tau_1}+\sqrt{\left(\beta u_M+\frac{1}{\tau_1}\right)^2+4u_M\left(\frac{\alpha}{\tau_2}-\frac{\alpha}{\tau_1}\right)}\right\}\right]^{1/2}. \tag{41f}$$

Boundary conditions and the joining conditions at the interface $r_1$ are:

$$x^1(R)=x^3(R)=0, \tag{42a}$$

$$\left.\frac{dx^1}{dr}\right|_{r=0}=\left.\frac{dx^3}{dr}\right|_{r=0}=0, \tag{42b}$$

$x^1$ : continuous at $r=r_1$ (42c)

$x^3$ : continuous at $r=r_1$, (42d)

$\frac{dx^1}{dr}$ : continuous at $r=r_1$, (42e)

$\frac{dx^3}{dr}$ : continuous at $r=r_1$. (42f)

$\Psi_2(R)=\Psi_4(R)=0$, (43a)

$\Psi_2$ : continuous at $r=r_1$, (43b)

$\Psi_4$ : continuous at $r=r_1$, (43c)

$\frac{d\Psi_2}{dr}$ : continuous at $r=r_1$, (43d)

$\frac{d\Psi_4}{dr}$ : continuous at $r=r_1$, (43e)

$\varphi(r_1)=0$. (43f)

Eight boundary conditions for $x^1$ and $x^3$ determine eight unknown constants, $A_1, B_1, C_1$ $D_1, A_2, B_2, C_2, D_2$. However, the seven boundary conditions are able to determine the eight unknown constants, $a_1, b_1, c_1, d_1, a_2, b_2, c_2$, and $d_2$, but not uniquely. As was the case for the one-zoned arrangement, the two-zoned reactor satisfies the necessary condition. In accordance with the engineering practice, the arrangement of high enrichment in inner zone naturally causes higher burnup of fuel in central zone, and such fact will lead to the unfavorable power peaking problem which cannot be acceptable in reactor design. Therefore, attempt is not made for the analysis of the two-zoned arrangement.

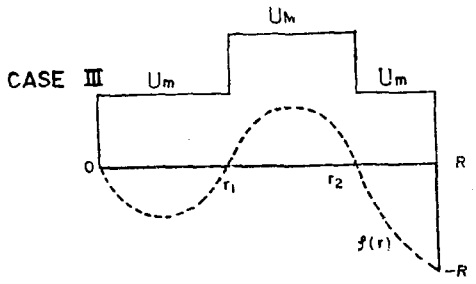


**Fig. 2. (c) three-zoned arrangement**

Case 3) : Three-zoned arrangement.

$(0\leq r\leq r_1)$

$$x_1=A_1S_1^1J_0(\mu_m r)+B_1S_1^2Y_0(\mu_m r)+C_1S_1^3I_0(\lambda_m r)+D_1S_1^4K_0(\lambda_m r), \quad (44a)$$

$$x^3=A_1J_0(\mu_m r)+B_1Y_0(\mu_m r)+C_1I_0(\lambda_m r)+D_1K_0(\lambda_m r), \quad (44b)$$

$(r_1\leq r\leq r_2)$

$$x^1=A_2S_2^1J_0(\mu_M r)+B_2S_2^2Y_0(\mu_M r)+C_2S_2^3I_0(\lambda_M r)+D_2S_2^4K_0(\lambda_M r), \quad (44c)$$

$$x_3=A_2J_0(\mu_M r)+B_2Y_0(\mu_M r)+C_2I_0(\lambda_M r)+D_2K_0)\lambda_M r), \quad (44d)$$

$(r_2\leq r\leq R)$

$$x^1=A_3S_3^1J_0(\mu_m r)+B_3S_3^2Y_0(\mu_m r)+C_3S_3^3I_0(\lambda_m r)+D_3S_3^4K_0(\lambda_m r), \quad (44e)$$

$$x^3=A_3I_0(\mu_m r)+B_3Y_0(\mu_m r)+C_3I_0(\lambda_m r)+D_3K_0(\lambda_m r). \quad (44f)$$

Substituting Equation (44) into Equation (25), the coupling constants $S_i^j$'s are obtained:

$$S_1^1=S_1^2=S_3^1=S_3^2=\tau_2(\mu_m^2+\beta u_m), \quad (44g)$$

$$S_1^3=S_1^4=S_3^3=S_3^4=-\tau_2(\lambda_m^2-\beta u_m), \quad (44h)$$

$$S_2^1=S_2^2=\tau_2(\mu_M^2+\beta u_M), \quad (44i)$$

$$S_2^3=S_2^4=-\tau_2(\lambda_M^2-\beta u_M), \quad (44j)$$

and

$(0\leq r\leq r_1)$

$$\Psi_2=a_1T_1^1J_0(\mu_m r)+b_1T_1^2Y_0(\mu_m r)+c_1T_1^3I_0(\lambda_m r)+d_1T_1^4K_0(\lambda_m r), \quad (45a)$$

$$\Psi_4=a_1J_0(\mu_m r)+b_1Y_0(\mu_m r)+c_1I_0(\lambda_m r)+d_1K_0(\lambda_m r), \quad (45b)$$

$(r_1\leq r\leq r_2)$

$$\Psi_2=a_2T_2^1J_0(\mu_M r)+b_2T_2^2Y_0(\mu_M r)+c_2T_2^3I_0(\lambda_M r)+d_2T_2^4K_0(\lambda_M r), \quad (45c)$$

$$\Psi_4=a_2J_0(\mu_M r)+b_2Y_0(\mu_M r)+c_2I_0(\lambda_M r)+d_2K_0(\lambda_M r), \quad (45d)$$

$(r_2\leq r\leq R)$

$$\Psi_2=a_3T_3^1J_0(\mu_m r)+b_3T_3^2Y_0(\mu_m r)+c_3T_3^3I_0(\lambda_m r)+d_3T_3^4K_0(\lambda_m r), \quad (45e)$$

$$\Psi_4=a_3J_0(\mu_m r)+b_3Y_0(\mu_m r)+c_3I_0(\lambda_m r)+d_3K_0(\lambda_m r). \quad (45f)$$

Boundary conditions and joining conditions at interfaces $r_1$ and $r_2$ are:

$$\frac{dx^1}{dr}\Big|_{r=0}=\frac{dx^3}{dr}\Big|_{r=0}=0, \quad (46a)$$

$x^1$, $x^3$, $\frac{dx^1}{dr}$, $\frac{dx^3}{dr}$ :

continuous at $r=r_1$, and $r=r_2$, (46b)

**Table 1. Selection of the optimal conditions**

| Type of Arrangement | No. of Unknown Constants | | | No. of Boundary Conditions | | | | Is the No. of Boundary Conditions Sufficient ? | Is It Optimal Arrangement ? |
|---|---|---|---|---|---|---|---|---|---|
| | $x^1, x^3$ | $\Psi_2, \Psi_4$ | Total | $x^1, x^3$ | $\Psi_2, \Psi_4$ | $\varphi$ | Total | | |
| One-zoned | 4 | 4 | 8 | 4 | 2 | 0 | 6 | No | No |
| Two-zoned | 8 | 8 | 16 | 8 | 6 | 1 | 15 | No | No |
| Three-zoned | 12 | 12 | 24 | 12 | 10 | 2 | 24 | Yes | Yes |
| Four-zoned | 16 | 16 | 32 | 16 | 14 | 3 | 33 | No (overdetermined) | No |
| Five-zoned | 20 | 20 | 40 | 20 | 18 | 4 | 42 | // | // |
| Six-zoned | 24 | 24 | 48 | 24 | 22 | 5 | 51 | // | // |
| Seven-zoned | 28 | 28 | 56 | 28 | 26 | 6 | 60 | // | // |

$$x^1(R)=x^3(R)=0, \tag{46c}$$

and

$$\Psi_2(R)=\Psi_4(R)=0,$$

$$\Psi_2,\ \Psi_4,\ \frac{d\Psi_2}{dr},\ \frac{d\Psi_4}{dr}:$$

continuous at $r=r_1$ and $r=r_2$, (47b)

$$\varphi(r_1)=\varphi(r_2)=0. \tag{47c}$$

Twelve boundary conditions for $x^1$ and $x^3$, and $\Psi_2$ and $\Psi_4$, respectively, determine all the unknown constants, that is to say that there exists a unique vector function $\Psi_i(i=0, 1, 2, 3, 4)$. In conclusion, therefore, the three-zoned arrangement satisfies Pontryagin's maximum principle and is eventually optimal.

In the same way, the analysis for other zoned arrangements more than 4 zones is quite possible considering the boundary conditions and joining conditions at interfaces [zeros of $\varphi(r)$]. As in Table 1, it comes to our notice that the number of zones increases as the number of boundary conditions increases. Due to too many boundary conditions, they overdetermine the solution, which means that solution does not exist at all in any case. Consequently, 4-, 5-, 6-, 7-zoned arrangements, etc., are considered not to be optimal according to the maximum principle. Therefore, the three-zoned arrangement is the most optimal type of fuel distribution, so far as the critical mass minimization problem is concerned.

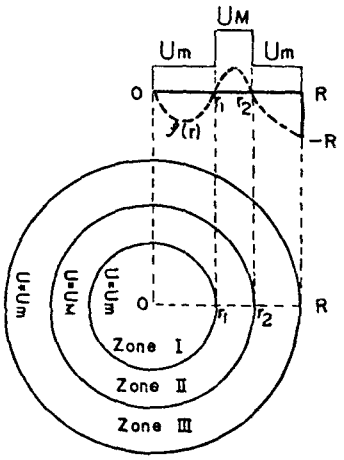


**Fig. 3. Distribution uranium enrichment in three-zoned reactor**

## 6. The Critical Condition

The general solutions (44) give the fast flux $x^1$ and the thermal flux $x^3$ in the three-zoned arrangement: zone I with the minimum value $u_m$ $(0\leq r\leq r_1)$; zone II with the maximum value $u_M(r_1\leq r\leq r_2)$; and zone III with the minimum value $u_m$ $(r_2\leq r\leq r_2)$. The flux distribution in each zone is the linear combination of the functions $J_0(r)$, $Y_0(r)$, $I_0(r)$ and $K_0(r)$. The quantities yet undetermined are the constants, $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3$, and $D_3$. These may be computed from the boundary conditions and the interfacial condition or joining conditions. From the physical point of view, the inferfacial conditions are the continuity of flux and net current both in

$$
\begin{aligned}
&S_1^1 J_0(\mu_m r_1)A_1 - S_2^1 I_0(\mu_M r_1)A_2 - S_2^2 Y_0(\mu_M r_1)B_2 + S_1^3 I_0(\lambda_m r_1)C_1 - S_2^3 I_0(\lambda_M r_1)C_2 - S_2^4 K_0(\lambda_M r_1)D_2 = 0\\
&-S_1^1 \mu_m I_1(\mu_m r_1)A_1 + S_2^1 \mu_M J_1(\mu_M r_1)A_2 + S_2^2 \mu_M (\mu_M r_1)B_2 + S_1^3 \lambda_m I_1(\lambda_m r_1)C_1 - S_2^3 \lambda_M I_1(\lambda_M r_1)D_2 = 0\\
&S_2^1 J_0(\mu_M r_2)A_2 - S_3^1 J_0(\mu_m r_2)A_3 + S_2^2 Y_0(\mu_M r_2)B_2 - S_3^2 Y_0(\mu_m \gamma_2)B_3 + S_2^3 I_0(\lambda_M r_2)C_2\\
&\qquad - S_3^3 I_0(\lambda_m r_2)C_3 + S_2^4 K_0(\lambda_M r_2)D_2 - S_3^4 K_0(\lambda_m r_2)D_3 = 0\\
&-S_2^1 \mu_M J_1(\mu_M r_2)A_2 + S_3^1 \mu_m J_1(\mu_m r_2)A_3 - S_2^2 u_M Y_1(\mu_M r_2)B_2 + S_3^2 \mu_m Y_1(\mu_m r_2)B_3\\
&\qquad + S_2^3 \lambda_M I_1(\lambda_M r_2)C_2 - S_3^3 \lambda_m I_1(\lambda_m r_2)C_3 - S_2^4 \lambda_M K_1(\lambda_M r_2)D_2 + S_3^4 \lambda_m K_1(\lambda_m r_2)D_3 = 0\\
&J_0(\mu_m r_1)A_1 - J_0(\mu_M r_1)A_2 - Y_0(\mu_M r_1)B_2 + I_0(\lambda_m r_1)C_1 - I_0(\lambda_M r_1)C_2 - K_0(\lambda_M r_1)D_2 = 0 \qquad (48)\\
&-\mu_m I_1(\mu_m r_1)A_1 + \mu_M J_1(\mu_M r_1)A_2 + \mu_M Y_1(\mu_M r_1)B_2 + \lambda_m I_1(\lambda_m r_1)C_1 - \lambda_M I_1(\lambda_M r_1)C_2 + \lambda_M K_1(\lambda_M r_1)D_2 = 0\\
&J_0(\mu_M r_2)A_2 - J_0(\mu_m r_2)A_3 + Y_0(\mu_M r_2)B_2 - Y_0(\mu_m r_2)B_3 + I_0(\lambda_M r_2)C_2\\
&\qquad - I_0(\lambda_m r_2)C_3 + K_0(\lambda_M r_2)D_2 - K_0(\lambda_m r_2)D_3 = 0\\
&-\mu_M J_1(\mu_M r_2)A_2 + \mu_m J_1(\mu_m r_2)A_3 - \mu_M Y_1(\mu_M r_2)B_2 + \mu_m Y_1(\mu_m r_2)B_3 + \lambda_M I_1(\lambda_M r_2)C_2\\
&\qquad - \lambda_m I_1(\lambda_m r_2)C_3 - \lambda_M K_1(\lambda_M r_2)D_2 + \lambda_m K_1(\lambda_m r_2)D_3 = 0\\
&S_3^1 J_0(\mu_M R)A_3 + S_3^2 Y_0(\mu_m R)B_3 + S_3^3 I_0(\lambda_m R)C_3 + S_3^4 K_0(\lambda_m R)D_3 = 0\\
&J_0(\mu_m R)A_3 + Y_0(\mu_m R)B_3 + I_0(\lambda_m R)C_3 + K_0(\lambda_m R)D_3 = 0
\end{aligned}
$$

the fast and thermal groups.

The application of conditions (46) to the general solutions (44) yields the following set of equations in the ten unknowns, $A_1, C_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3$ and $D_3$, where the boundary conditions (46) give $B_1=0$ and $D_1=0$.

The above set consists of ten simultaneous algebraic equations with ten unknown constants. Since these equations are homogeneous, it is quite impossible to obtain the absolute values for all the ten quantities and, as in all the steady-state reactor problems, one constant always remains undetermined there. When it is possible to specify the shape of the flux in a critical reactor, its magnitude depends upon the operating power of the system and is not determined by the group equation. The only nontribial solution of the above simultaneous equation requires that determinant of the coefficients of $A_1$, $C_1$, $A_2$, $B_2$, $C_2$, $D_2$, $A_3$, $B_3$, $C_3$, and $D_3$ vanishes in accordance with Cramer's rule, Eq. (49).

This expression may be regarded as the criticality condition (the conditon that there should exist a steady flux) in the two-group approximation. In the usual way, the specification of the fuel enrichment allows the determination of the core size, or vice versa, through the above equation. The prime importance in this paper is to determine the fuel enrichment of the critical mass when the shape and dimensions of the reactor are specified and composition of the nonfuel components in the core is given. The calculation for the determination of this problem may be carried out by assuming various fuel loadings of different enrichment and finally selecting the very fuel enrichment that satisfies all the system requirements determined above.

## 7. The Numerical Calculation

The numerical calculation to solve the given problem is performed by a digital computer. In order to obtain $u_m$ and $u_M$ of the uranium satisfying the determinant to be zero, it is absolutely necessary to iterate the computation procedure. Once the two values for the fuel enrichment, $u_m$ and $u_M$, are assumed, and all the nuclear parameters are computed, then the corresponding values to $\mu_m^2$, $\mu_M^2$, $\lambda_m^2$

$$
\begin{vmatrix}
S_1^1 J_0(\mu_m r_1) & -S_2^1 J_0(\mu_M r_1) & 0 & -S_2^2 Y_0(\mu_M r_1) & 0 & S_1^3 I_0(\lambda_m r_1) & -S_2^3 I_0(\lambda_M r_1) & 0 & -S_2^4 K_0(\lambda_M r_1) & 0 \\
-S_1^1 \mu_m J_1(\mu_m r_1) & S_2^1 \mu_M J_1(\mu_M r_1) & 0 & S_2^2 \mu_M Y_1(\mu_M r_1) & 0 & S_1^3 \lambda_m I_1(\lambda_m r_1) & -S_2^3 \lambda_M I_1(\lambda_M r_1) & 0 & S_2^4 \lambda_M K_1(\lambda_M r_1) & 0 \\
0 & S_2^1 J_0(\mu_M r_2) & -S_3^1 J_0(\mu_m r_2) & S_2^2 Y_0(\mu_M r_2) & -S_3^2 Y_0(\mu_m r_2) & 0 & S_2^3 I_0(\lambda_M r_2) & -S_3^3 I_0(\lambda_m r_2) & S_2^4 K_0(\lambda_M r_2) & -S_3^4 K_0(\lambda_m r_2) \\
0 & -S_2^1 \mu_M J_1(\mu_M r_2) & S_3^1 \mu_M J_1(\mu_M r_2) & -S_2^2 \mu_M Y_1(\mu_M r_2) & S_3^2 \mu_m Y_1(\mu_m r_2) & 0 & S_2^3 \lambda_M I_1(\lambda_M r_2) & -S_3^3 \lambda_m I_1(\lambda_m r_2) & -S_2^4 \lambda_M K_1(\lambda_M r_2) & S_3^4 \lambda_m K_1(\lambda_m r_2) \\
J_0(\mu_m r_1) & -J_0(\mu_M r_1) & 0 & -Y_0(\mu_M r_1) & 0 & I_0(\lambda_m r_1) & -I_0(\lambda_M r_1) & 0 & -K_0(\lambda_M r_1) & 0 \\
-\mu_m J_1(\mu_m r_1) & \mu_M J_1(\mu_M r_1) & 0 & \mu_M Y_1(\mu_M r_1) & 0 & \lambda_m I_1(\lambda_m r_1) & -\lambda_M I_1(\lambda_M r_1) & 0 & \lambda_M K_1(\lambda_M r_1) & 0 \\
0 & J_0(\mu_M r_2) & -J_0(\mu_m r_2) & Y_0(\mu_M r_2) & -Y_0(\mu_m r_2) & 0 & I_0(\lambda_M r_2) & -I_0(\lambda_m r_2) & K_0(\lambda_M r_2) & -K_0(\lambda_m r_2) \\
0 & -\mu_M J_1(\mu_M r_2) & \mu_m J_1(\mu_m r_2) & -\mu_M Y_1(\mu_M r_2) & \mu_M I_1(\mu_M r_2) & 0 & \lambda_M I_1(\lambda_M r_2) & -\lambda_m I_1(\lambda_m r_2) & -\lambda_M K_1(\lambda_M r_2) & \lambda_m K_1(\lambda_m r_2) \\
0 & 0 & S_3^1 J_0(\mu_M R) & 0 & S_3^2 Y_0(\mu_m R) & 0 & 0 & S_3^3 J_0(\lambda_m R) & 0 & S_3^4 K_0(\lambda_m R) \\
0 & 0 & J_0(\mu_m R) & 0 & Y_0(\mu_m R) & 0 & 0 & I_0(\lambda_m R) & 0 & K_0(\lambda_m R)
\end{vmatrix} = 0 \quad (49)
$$

and $\lambda_M^2$ may be obtained from the relation (38e), (38f), (41e), (41f).

These values, along with the appropriate data for the various functions, $J_o$, $Y_o$, $I_o$, and $K_o$, may then be inserted in the determinant (49). Should the value of the determinant be found to be identically zero, the assumed fuel enrichment, in fact, is, the critical mass enrichment. If the value turns out to be other than zero, the compution should be repeated until this condition comes to be satisfied. It should be recognized, however, that, in this procedure, it is required to recompute the nuclear parameters corresponding to the fuel enrichment. Nevertheless, since these parameters are to be independent of uranium enrichment in the original assumption, it is not necessary to make adjustments for these quantities.

With all these factors in mind, numerical calculation is carried out for the case of Kori reactor geometry, of which dimension and other parameters are given in Table 2. As described in the Table 2, fast and thermal diffusion coefficients for Kori reactor are 1.367 cm and 0.2294 cm, respectively, whereas the macroscopic cross-section is 0.03022 $cm^{-1}$. So far as the atomic density is concerned, that of uranium which consists of U-235 atoms and U-238 atoms is a third of water and about a half of oxygen atoms. The radii of each zone, $r_1, r_2$ and R, are 70.76 cm, 100.01 cm and 122.6 cm.

In order for a reactor to be just critical, the ten-by-ten determinant must turn out to be zero. As was previously made clear, the minimum enrichment depends on the maximum enrichment, or vice versa, providing that the dimension of the three-zoned reactor, such as $r_1, r_2$ and $R$, is given. Conversely, $r_1$ and $r_2$ are mutually dependent upon each other.

In this calculation, the minimum enrichment

**Table 2. Reactor constants**

| | |
|---|---|
| Diffusion coefficient, D, (cm) | 1.367(fast)<br>0.2294(thermal) |
| Macroscopic cross-section, $\Sigma_1$, [$cm^{-1}$] (absorption+scattering) | 0.03922(fast) |
| Atomic density(atoms/$cm^3$) | |
| Uranium; $N^U (= N^{235} + N^{238})$ | $6.994\times10^{21}$ |
| Water; $N^{H2O}$ | $1.947\times10^{22}$ |
| O(in $UO_2$); $N^O$ | $1.399\times10^{22}$ |
| Radius, (cm) | |
| Zone 1; $r_1$ | 70.76 |
| Zone 2; $r_2$ | 100. 1 |
| Zone 3; R | 122. 6 |

is determined for a given maximum enrichment $u_M$ so that the reactor goes critical when $r_1, r_2$ and reactor radius are given. The geometrical and nuclear data referred to herein are quoted from PSAR[16] for the Kori Nuclear Power Plant, APEX-369 and others[15, 17, 18], and the details are listed in Table 2. In order to bring forth the results as accurate as possible, the numerical calculation is repeated, iterating the steps between the parameters ($\mu_m$, $\mu_M$, $\lambda_m$ and $\lambda_M$) and the determinant calculation.

Fig. 4 shows the computed results listed in several pairs of the minimum-and-maximum-enrichment combination making the concerned

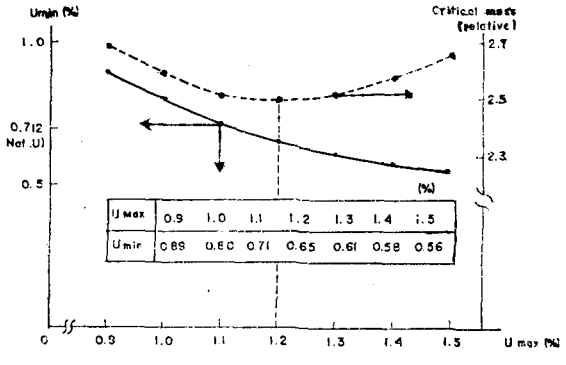


**Fig. 4. Maximum *vs.* minimum enrichment**
(Solid line represents the combination of the maximum and minimum enrichment, while the dotted line represents the relative critical mass for each case)

determinant zero. It is possible from this relationship to find the maximum or minimum enrichment which makes the minimum critical mass in accordance with the derived theory so far, when the ratio between the maximum or minimum enrichment is given, or when either the maximum or minimum enrichment is given in advance.

It is more than clear as described in Fig. 4 that the minimum enrichment of uranium gradually decreases as the maximum enrichment increases. For instance, the latter is 0.89% when the former is set as 0.9%, while the latter is 0.80 when the former is given as 1.0. On the other hand, the minimum enrichment turns out to be 0.71 if the maximum enrichment is 1.1. In the same manner, the minimum enrichments are calculated to be 0.65, 0.61, 0.58 and 0.56% if and when the maximum enrichments are given in advance as 1.2, 1.3, 1.4 and 1.5%, respectively.

Since the critical mass is obtainable from the relationship which is comprised of pairs of the maximum and minimum enrichment, it is able to find the least possible critical mass among all the enrichment combinations in the Fig. 4, and it is found to be the case of 1.2% for the maximum enrichment while 0.65% for the minimum enrichment, the reason being that, so far as the relative critical mass is concerned, this particular case is most satisfactory. From the theoretical point of view, and apart from the availability of such enriched uranium, the combination of the 1.2% maximum enrichment in the middle zone and 0.65% minimum enrichment in inner and outer zones is the most optimistic condition for the fuel loading. Nevertheless, because the critical mass problem cannot be the only factor to be considered in reactor design, this problem must be correlated with other factors such as power peaking, neutron flux distribution, fuel economy, safety aspects,

etc., so that the compound optimization might be sought for in reasonable and logical way.

Mention must be made of the fact that, when a power reactor reaches the end of its cycle, many factors give negative effect to the reactivity so that the critical mass eventually increases more than the case of clean cold condition.

The factors which are associated with the negative reactivity in the reactor at the end of its cycle are as follows (in the case of Kori reactor)[16,19,20]:

1. Control rods
2. Fission products, especially xenon and samarium
3. Parastic capture by structural materials such as cladding
4. Residual boric acid in the coolant which is less than 10 ppm.
5. Doppler effect
   ($-1.0\times10^{-5}$ to $-1.6\times10^{-5}$ $\triangle$k/k/g/cm$^3$)
6. Moderator temperature coefficient
   ($0.3\times10^{-4}$ to $-3.5\times10^{-4}$ $\triangle$k/k/$^o$F)
7. Moderator pressure coefficient
   ($-0.3\times10^{-4}$ to $+3.5\times10^{-6}$ $\triangle$k/k/psi)
8. Moderator density coefficient
   ($-1.10$ to $+0.30$ $\triangle$k/k/g/cm$^3$)
9. Fuel depletion
10. Vertical leakage of neutrons.

The respective analysis of each component listed above is beyond the scope of this work; however, in the actual design of a reactor, all the effects like the above should be given thorough consideration so that the minimum critical mass at the beginning of a fuel cycle could be derived from the minimum critical mass calculated in this paper. In such a sense, the result of this work can be said to have paved the way toward the calculation of correct critical mass of a power reactor so as to optimize the fuel loading at the beginning of fuel cycle.

It is assumed at the beginning that $\Sigma_2$ is a function of uranium enrichment and is not enrichment itself, on the ground that:

$$\Sigma_2=(\sigma_a^{235}N^{235}+N^{others}\ \sigma_a^{others})=(\gamma u'+\delta)=u.$$

In the actual computer calculation, the values of $\gamma$ and $\delta$ come out to be as follows:

$$\gamma=\sigma_a^{235}(N^{235}+N^{238})=4.20077,$$

$$\delta=N^{others}\ \sigma_a^{others}=0.0280587.$$

It is programmed upon the calculation that the maximum enrichment is set as 1.6% at first and then it is gradually reduced step-by-step-wise (0.1% each per step) until the determinant turns out to be zero, so that it corresponds to the optimal combination of the maximum and minimum enrichment.

Since it is practically impossible to find the zero value by solving the determinant, it is assumsed that the zero value must exist somewhere between the plus and minus values. Therefore, the actual enrichment which makes the value of the determinant zero is sought by means of interpolating the computed two values (in this case, one is plus and the other is minns).

All other constants such as $\lambda$ and $\mu$ as well as coupling constants are also calculated with the help of computer. On the other hand, however, the Bessel functions are simply quoted from the computer library. The computation is actually iterated dozens of score times to find each zero value. For the computer program, please refer to the Fig. 5.

## 8. Conclusion

With the application of Pontryagrn's maximum principle to the two-group equation, a Hamiltonian is derived for a reactor model of cylindrical geometry which is assumed to have the physical dimension of Kori reactor. Then the linear dependence of $u(r)$ on the Hamiltonian comes to imply that $u(r)$ is a step function. This linear dependence is ascribed to the fact that only the macroscopic absorption

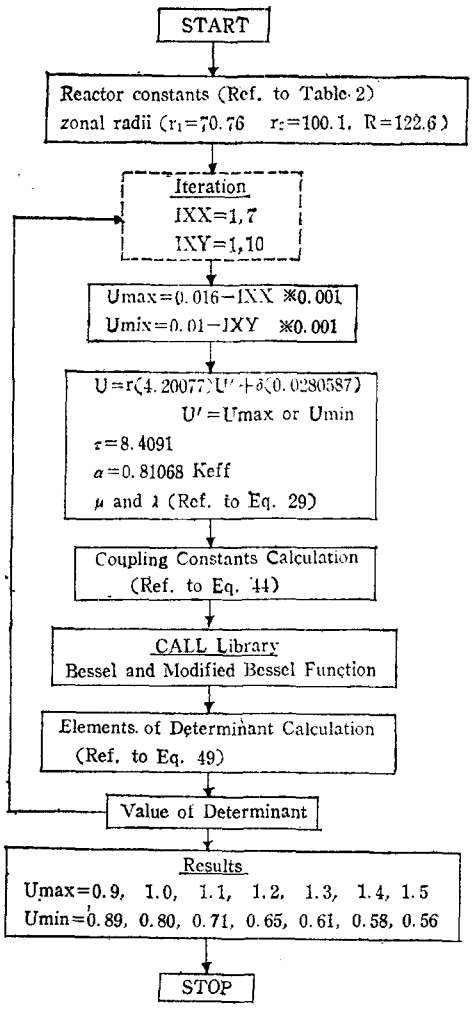


**Fig. 5. Simplified flow chart for calculation procedure**

cross-section of $U^{235}$ nuclides is linearly dependent upon the enrichment and other nuclear parameters do not depend on it. It is clear that the strict assumption of all the nuclear parameters may complicate the control function $u(r)$.

Unlike the previous works by others[1-6], the range of the minimum enrichment and that of the maximum enrichment are not limited in this paper; moreover, the range is determined from the critical condition, that is to say, it is determined from the critical determinant. It is proved that the theory developed herein is logical on the ground that the computed results satisfy the necessary and sufficient conditions of the theoretical analysis. From the numerical calculation, several pairs of the minimum-and-maximum enrichment combination are obtained, which all make the determinant zero, *i.e.*, that satisfies the critical condition so that the reactor in question is able to go critical. Among all the calculated values, the least possible critical mass is found to be the case of 1.2% maximum enrichment and for the middle zone and 0.65% minimum enrichment for the inner and outer zones. For more accurate analysis of this problem it is recommendable to extend the scope of work so that the consideration of vertical axis may be taken into account and the bounded power density problem may be treated as well.

When and if these factors be made into being, the optimization problem of other subjects, such as power maximization, burn-up maximization, refueling optimization, and the like, could be conveniently achieved. It is suggested as a further work in the future that the zonal radii may be subject to changes of the preset enrichment, or conversely, the maximum enrichment or the minimum enrichment may be set as a variable versus the radius.

Since the reactor model referred to herein is simply quoted from the dimension of Kori reactor, all the nuclear design parameters are not quantitatively analysed in the treatment, though they are considered in lumped-sum effect. In other words, since the approach in this paper is based on the clean cold condition of the initial reactor, the calculation of the minimum critical mass by means of the maximum principle for the case of Kori reactor model does not mean that the engineering design data could be accurately resulted from this analysis.

## Acknowledgement

The author's special thank goes to Professor Chol Kon Chee under whose supervision this work is made successful. He is also indebted to Professors Heung Suk Yang, Myoung Sam Ko and Young Moon Park of the Department of Electrical Engineering and Professor Chang-Hyun Chung of the Department of Nuclear Engineering, Seoul National Uniuersity, for their discussion and encouragement. The author wishes to acknowledge the tireless assistance provided by Mr. Sang Won Kim who has participated in this project from the beginning till the correction and printing of the paper. His appreciation is extended to Messrs. Jean Soo Kim, Ji Bok Lee, Sang Keun Lee and Sam Kon Kim who are all involved in discussions as well as programming and numerical calculation of the problem by Control Data's digital computer, Cyber 70, being installed in the Korea Institute of Science & Technology.

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