Sequential Sampling Estimation for True Population on the Stratified Sampling

Chun Ho Choi*

1. Introduction

While designing a sampling survey a list of the past population is sometimes used as a frame of random sampling. But the sampling in the case is conducted on the basis of the past data. As the results, units that did not exist at the time of sample taking may be included while new units are excluded from the list of population. Accordingly, when it is attempted to estimate a certain character of the true population under this situation, it is expected that a much more accurate estimation may be obtained by paying a close attention to the lost units as well as new units.

In this paper, an attempt is made to introduce a new method of figuring out the lost units as well as new units and then, on the basis of this new method, to deduce estimation of variance, optimum allocation, etc, of one-stage, two-stage and three-stage stratified samplings.

2. The Case When the New Added Units are Known in the q-th Sampling

Let π_0 be an initial list of a population. In the q-th sampling, let the list of true population be π_q(unknown), $D_{q-1}=\pi_0 \cup \pi_1 \cup \cdots\cdots \cup \pi_{q-1}$ be the sum of the previous list of a population(known), and A_q be the list of newly added units in q-th sampling (known), then, C_q is a set of all absent in the q-th sampling, when it is defined in the following manner:

$B_q=\pi_q \cap D_{q-1}$, $C_q=D_{q-1}-B_q$ (unknown).

Obviously $\pi_q=A_q \cup B_q$ and

$D_{q-1}=B_q \cup C_q$. Let $A_q=\{\theta_1, \theta_2, \ldots\ldots, \theta_M\}$, $B_q=\{w_1, w_2, \ldots\ldots, w_K\}$, and $C_q=\{w_{K+1}, \ldots\ldots, w_N\}$, where M, N, A_q and D_{q-1} are known, but K is unknown. We consider the following

*Professor of Mathematics, Hanyang University.

random variables x_i and y_j defined on A_q and D_{q-1}, respectively. If $\theta_i \in A_q$ then $x(\theta_i)$ is a value of character α under the consideration and if $w \in D_{q-1}$, then $y(w)$ has a non-zero value except $y(w)=0$ for $w \in C_q$.

3. One Stage Stratified Sampling in Sequential Procedures.

We divided population into L nonoverlapping subpopulations and called them strata. In the the q-th sampling of sequential procedures N elements in D_{q-1} and M elements in A_q can be divided into above L strate, respectively. Let elements of D_{q-1} in the i-th stratum be w_{i1}, $w_{i2}, \ldots\ldots, w_{iK_i}$, $w_{iK_{i+1}}, \ldots\ldots, w_{iN_i}$, where $w_{i1}, \ldots\ldots, w_{iK_i} \in B_q$, $w_{iK_{i+1}}, \ldots\ldots, w_{iN_i} \in C_q$, and let elements of A_q in i-th stratum be θ_{i1}, $\theta_{i2}, \ldots\ldots, \theta_{iM_i}$. Let the value of a character α of these again be $y_{ij}=y(w_{ij})$ for $j=1, \ldots\ldots, K_i$, $y_{ij}=y(w_{ij})=0$ for $j=K_{i+1}, \ldots\ldots, N_i$, and $x_{ij}=x(\theta_{ij})$ for $j=1, \ldots\ldots, M_i$. Obviously $N=\sum_{i=1}^{L} N_i$, $M=\sum_{i=1}^{L} M_i$.

In the i-th stratum, let total value of a character α in B_q, A_q and D_{q-1} be $Y_i=\sum_{j=1}^{K_i} y_{ij}$, $X_i=\sum_{j=1}^{M_i}$, $Z_i=\sum_{j=1}^{N_i} y_{ij}=\sum_{j=1}^{K_i} y_{ij}=Y_i$, their means be $\overline{Y}_i=\frac{Y_i}{K_i}$, $\overline{X}_i=\frac{X_i}{M_i}$, $\overline{Z}_i=\frac{Y_i}{N_i}$, respectively, and total value of a character in the i-th stratum of true population π_q be $T_i=\sum_{j=1}^{K_i} y_{ij}+\sum_{j=1}^{M_i} x_{ij}=Y_i+X_i$. In addition, the total value of true population π_q is defined to be $T=\sum_{i=1}^{L}(X_i+Y_i)=\sum_{i=1}^{L}\sum_{j=1}^{M_i} x_{ij}+\sum_{i=1}^{L}\sum_{j=1}^{K_i} y_{ij}$. Now, we draw a sample from each stratum independently with equal probability. Let the sample size in the i-th stratum for D_{q-1} and A_q be n_i and m_i, respectively, and let n_i samples be y_{i1}', $y_{i2}' \ldots\ldots y_{ik_i}'$ (belong to B_q), $y_{ik_i+1}'=y_{ik_i+2}'=\ldots\ldots=y_{in_i}'=0$ (belong to C_q), and m_i samples be $x_{i1}', x_{i2}' \ldots\ldots x_{im_i}'$ (belong to A_k).

Theorem 3-1 In the q-th sampling of sequential procedures

(1) $\hat{T}=\sum_{i=1}^{L}\left(\frac{M_i}{m_i}\sum_{j=1}^{m_i} x_{ij}'+\frac{N_i}{n_i}\sum_{j=1}^{n_i} y_{ij}'\right)$ is unbiased estimate of total $T=\sum_{i=1}^{L} T_i$ of the true popuation π_q.

(2) The variane of $\hat{T}$ is

$$V(\hat{T})=\sum_{i=1}^{L}\left[M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\frac{\sigma_{1i}^2}{m_i}+N_i^2\left(\frac{N_i-n_i}{N_i-1}\right)\frac{\sigma_{2i}^2}{n_i}\right]$$

(3) Unbiased estimate of $V(\hat{T})$ is

$$\hat{V}(\hat{T})=\sum_{i=1}^{L}\left[M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\frac{S_{1i}^2}{m_i}+N_i^2\left(\frac{N_i-n_i}{N_i-1}\right)\frac{S_{2i}^2}{n_i}\right]$$

where

$$\sigma_{1i}^2=\frac{1}{M_i}\sum_{j=1}^{M_i}(x_{ij}-\bar{x}_i)^2$$

$$\sigma_{2i}^2=\frac{1}{N_i}\sum_{j=1}^{N_i}(y_{ij}-\bar{y}_i)^2=\frac{1}{N_i}\left[\sum_{j=1}^{K_i}\left(y_{ij}-\frac{Y_i}{N_i}\right)^2+\frac{(N_i-K_i)}{N_i^2}x_i^2\right]$$

$$S_{1i}^2=\frac{1}{m_i-1}\sum_{j=1}^{m_i}(x_{ij}'-\bar{x}_i')^2$$

$$S_{2i}^2=\frac{1}{n_i-1}\left[\sum_{j=1}^{k_i}\left(y_{ij}'-\frac{1}{n_i}\sum_{j=1}^{k_i}y_{ij}'\right)^2+\left(\frac{n_i-k_i}{n_i^2}\right)\left(\sum_{j=1}^{k_i}x_{ij}'\right)^2\right]$$

⟨Proof⟩ Since $\hat{T}=\sum_{i=1}^{L}\left(\frac{M_i}{m_i}\sum_{j=1}^{m_i}x_{ij}'+\frac{N_i}{n_i}\sum_{j=1}^{n_i}y_{ij}'\right)$

so that

$$E(\hat{T})=\sum_{i=1}^{L}M_iE\left(\frac{1}{m_i}\sum_{j=1}^{m_i}x'_{ij}\right)+\sum_{i=1}^{L}N_iE\left(\frac{1}{n_i}\sum_{j=1}^{k_i}x_{ij}'\right)$$

$$=\sum_{i=1}^{L}M_i\bar{X}_i+\sum_{i=1}^{L}N_i\bar{Y}_i=\sum_{i=1}^{L}X_i+\sum_{i=1}^{L}Y_i=T.$$

This proves (1).

Since $(x_{i1}', x_{i2}' x_{im_i}')$ and $(y_{i1}', y_{i2}', \ldots\ldots, y_{in_i}')$ are mutually independent, so that

$$V(\hat{T})=V\left[\sum_{i=1}^{L}\left(\frac{M_i}{m_i}\sum_{j=1}^{m_i}x_{ij}'+\frac{N_i}{n_i}\sum_{j=1}^{k_i}y_{ij}'\right)\right]$$

$$=\sum_{i=1}^{L}M_i^2V\left(\frac{1}{m_i}\sum_{j=1}^{m_i}x_{ij}'\right)+\sum_{i=1}^{L}N_i^2V\left(\frac{1}{n_i}\sum_{j=1}^{k_i}y_{ij}'\right)$$

$$=\sum_{i=1}^{L}M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\frac{\sigma_{1i}^2}{m_i}+\sum_{i=1}^{L}N_i^2\left(\frac{N_i-n_i}{N_i-1}\right)\frac{\sigma_{2i}^2}{n_i}$$

This proves (2).

Since

$$\frac{1}{n_i-1}\sum_{j=1}^{n_i}(y_{ij}'-\bar{y}_i')^2=\frac{1}{n_i-1}\left[\sum_{j=1}^{k_i}(y_{ij}'-\bar{y}_i')^2+\sum_{j=k_i+1}^{n_i}(y_{ij}'-\bar{y}_i')^2\right]$$

$$=\frac{1}{n_i-1}\left[\sum_{j=1}^{k_i}(y_{ij}'-\bar{y}_i')^2+\left(\frac{n_i-k_i}{n_i^2}\right)\left(\sum_{j=1}^{k_i}y_{ij}'\right)^2\right]$$

$$=S_{2i}^2,$$

so that $E(S_{1i}^2)=\sigma_{1i}^2$, $E(S_{2i}^2)=\sigma_{2i}^2$, so that

$$E(\hat{V}(\hat{T}))=\sum_{i=1}^{L}M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\frac{\sigma_{1i}^2}{m_i}+\sum_{i=1}^{L}N_i^2\left(\frac{N_i-n_i}{N_i-1}\right)\frac{\sigma_{2i}^2}{n_i}$$

Theorem 3-2 In the q-th sampling of sequential procedures, the variance of the estimated total $\hat{T}$ in theorem 3-1, is the minimum when m_i and n_i are proportional to $\frac{M_i\sigma_{1i}}{\sqrt{C_{1i}}}$ and $\frac{N_i\sigma_{2i}}{\sqrt{C_{2i}}}$ respectively, where C_{1i} and C_{2i} are the cost per unit sampling from

A_q and D_{q-1} in the i-th stratum.

⟨Proof⟩ Let C_0 represent an overhead cost.

Then the total cost is $C=C_0+\sum_{i=1}^{L}(C_{1i}m_i+C_{2i}n_i)$

Since

$$V(\hat{T})=\sum_{i=1}^{L}\left[M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\frac{\sigma_{1i}^2}{m_i}+N_i^2\left(\frac{N_i-n_i}{N_i-1}\right)\frac{\sigma_{2i}^2}{n_i}\right]$$

the problem is to minimize $(V(\hat{T})$ with the restriction of cost C. We use the method of Lagrangerian multipriers and select the m_i and n_i to be minimized:

$$f_1(m_i,n_i)=\sum_{i=1}^{L}\left[M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\frac{\sigma_{1i}^2}{m_i}+N_i^2\left(\frac{N_i-n_i}{N_i-1}\right)\frac{\sigma_{2i}^2}{n_i}\right]$$
$$+\lambda\left[c_0+\sum_{i=1}^{L}(c_{1i}m_i+c_{2i}n_i)\right]$$

$$\frac{\partial f}{\partial m_i}=\sum_{i=1}^{L}\left[-\frac{M_i^3\sigma_{1i}^2}{(M_i-1)m_i^2}+\lambda c_{1i}\right]=0 \quad \therefore\ m_i^2\fallingdotseq\frac{M_i^2\sigma_{1i}^2}{\lambda c_{1i}}\ (M_i-1\fallingdotseq M_i)$$

$$\frac{\partial f}{\partial n_i}=\sum_{i=1}^{L}\left[-\frac{N_i^3\sigma_{2i}^2}{(N_i-1)n_i^2}+\lambda c_{2i}\right]=0 \quad \therefore\ n_i^2\fallingdotseq\frac{N_i^2\sigma_{2i}^2}{\lambda c_{2i}}\ (N_i-1\fallingdotseq N_i)$$

when $M_i-1\fallingdotseq M_i$, $N_i-1\fallingdotseq N_i$, $m_i:n_i\fallingdotseq\frac{M_i\sigma_{1i}}{\sqrt{c_{1i}}}:\frac{N_i\sigma_{2i}}{\sqrt{c_{2i}}}$

If total size of sampling for i-th stratum is given, say n', then we can get

$$m_i=\frac{n'M_i\sigma_{1i}}{\sqrt{c_{1i}}\left(\frac{M_i\sigma_{1i}}{\sqrt{c_{1i}}}+\frac{N_i\sigma_{2i}}{\sqrt{c_{2i}}}\right)} \quad\text{and}\quad n_i=\frac{n'N_i\sigma_{2i}}{\sqrt{c_{2i}}\left(\frac{M_i\sigma_{1i}}{\sqrt{c_{1i}}}+\frac{N_i\sigma_{2i}}{\sqrt{c_{2i}}}\right)}$$

4. Two-stage Stratified Sampling in Sequential Procedures

The population is divided into L nonoverlapping subpopulations called strata, and the i-th stratum has M_i primary sampling units (P. S. U.) E_{i1}, E_{i2}, ...$E_{iM_i}(i=1,......L)$. E_{ij} is again divided into nonoverlapping stratum. In q-th sampling of sequential procedures, N elements in D_{q-1} and M elements in A_q can be divided into L strata as in the above, respectively, and i-th stratum has M_i P. S. U. E_{ij} is also again divided into nonoverlapping stratum w_{ij_1}, ...$w_{ijK_{ij}}$ (belong to B_q), $w_{ij(K_{ij}+1)}$, ...$w_{ijN_{ij}}$ (belong to C_q) and θ_{ij1}, ...$\theta_{ijM_{ij}}\in A_q$, respectively. Let the value of a character α in clusters be $y_{ijk}=y(w_{ijk})$ for $k=1$, ...K_{ij}, $y_{ijk}=y(w_{ijk})=0$ for $k=K_{ijk}+1$, ...N_{ij} and $x_{ijk}=x(\theta_{ijk})$ for $k=1$, ...M_{ij}. Now, at each stratum, primary sampling is made independently. At the i-th stratum, a sample of size M_i is drawn with equal prabability and let E_{i1}, E_{i2}, ...

E_{iM_i} denote the P. S. U. drawn.

At the second stage, from each of P. S. U's $E_{ij}(i=1,\ldots\ldots L,\ j=1,\ldots\ldots M_i)$ is drawn independently, in which D_{q-1} and A_q are included, respectively. From $E_{ij}(i=1,\ldots\ldots L,\ j=1,\ldots\ldots M_i)$ in D_{q-1} we draw sample size n_{ij} with equal probability and observe the number of elements and let it be $y_{ij1}'\ldots\ldots y_{ijk_{ij}}'$ (belong to B_q), $y_{ijk_{ij+1}}'=y_{ijk_{ij+2}}'=\ldots\ldots=y_{ijn_{ij}}'=0$ (belong to C_q). Similary from E_{ij} $(i=1\ldots\ldots L, j=1\ldots\ldots M_i)$ in A_q we draw samples of size m_{ij} with equal probability and let m_{ij} samples be $x_{ij_1}', x_{ij_2}'\ldots\ldots x_{ijm_{ij}}'$.

Theorem 4-1: At the q-th sampling of sequential procedures,

(1) $$\hat{T}=\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}x_{ijk}'+\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{N_{ij}}{n_{ij}}\sum_{k=1}^{k_{ij}}y_{ijk}'$$

is an unbiased estimate of the total

$$T=\sum_{i=1}^{L}(X_i+Y_i)=\sum_{i=1}^{L}\sum_{j=1}^{M_i}\left(\sum_{k=1}^{M_{ij}}x_{ijk}+\sum_{k=1}^{k_{ij}}y_{ijk}\right)$$

(2) The variance of $\hat{T}$ is

$$V(\hat{T})=\sum_{i=1}^{L}\left[M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\left(\frac{\sigma_{1i}^2+\sigma_{2i}^2}{m_i}\right)+\frac{M_i}{m_i}\sum_{j=1}^{M_i}M_{ij}^2\left(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\right)\frac{\sigma_{1ij}^2}{m_{ij}}\right.$$
$$\left.+\frac{M_i}{m_i}\sum_{j=1}^{M_i}N_{ij}^2\left(\frac{N_{ij}-n_{ij}}{N_{ij}-1}\right)\frac{\sigma_{2ij}^2}{n_{ij}}\right]$$

where

$$\sigma_{1ij}^2=\frac{1}{M_{ij}}\sum_{k=1}^{M_{ij}}(x_{ijk}-\bar{x}_{ij})^2,\quad \bar{x}_{ij}=\frac{1}{M_{ij}}\sum_{k=1}^{M_{ij}}x_{ijk},$$

$$\sigma_{2ij}^2=\frac{1}{N_{ij}}\left[\sum_{k=1}^{K_{ij}}\left(y_{ijk}-\frac{Y_{ij}}{N_{ij}}\right)^2+\left(\frac{N_{ij}-K_{ij}}{N_{ij}^2}\right)Y_{ij}^2\right]$$

$$Y_{ij}=\sum_{k=1}^{N_{ij}}y_{ijk}=\sum_{k=1}^{K_{ij}}y_{ijk}$$

(3) Unbiased estimate of $V(\hat{T})$ is

$$\hat{V}(\hat{T})=\sum_{i=1}^{L}\left[M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\left(\frac{S_{1i}^2+S_{2i}^2}{m_i}\right)+\frac{M_i}{m_i}\sum_{j=1}^{M_i}M_{ij}^2\left(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\right)\frac{S_{1ij}^2}{m_{ij}}\right.$$
$$\left.+\frac{M_i}{m_i}\sum_{j=1}^{M_i}N_{ij}^2\left(\frac{N_{ij}-n_{ij}}{N_{ij}-1}\right)\frac{S_{2ij}^2}{n_{ij}}\right]$$

where

$$S_{1ij}^2=\frac{1}{m_{ij}-1}\sum_{k=1}^{m_{ij}}(x_{ijk}'-\bar{x}_{ij}')^2$$

$$S_{2ij}^2=\frac{1}{n_{ij}-1}\left[\sum_{k=1}^{k_{ij}}\left(y_{ijk}'-\frac{y_{ij}'}{n_{ij}}\right)^2+\left(\frac{n_{ij}-k_{ij}}{n_{ij}^2}\right)y_{ij}^2\right]$$

$$\bar{x}_{ij}'=\frac{1}{m_{ij}}\sum_{k=1}^{m_{ij}}x_{ijk}',\quad \bar{y}_{ij}'=\frac{1}{n_{ij}}\sum_{k=1}^{k_{ij}}y_{ijk}'$$

$$y_{ij}'=\sum_{k=1}^{k_{ij}} y_{ijk}'$$

〈Proof (1)〉 Let $\underset{i}{E}$ denote expectation of the first stage sampling and $\underset{j}{E}^{(i)}$ dentotoe expectation of the second stage sampling when the first stage sampling is fixed.

$$E(\hat{T})=E\Big[\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}x_{ijk}'+\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{N_{ij}}{n_{ij}}\sum_{k=1}^{k_{ij}}y_{ijk}'\Big]$$

$$=\sum_{i=1}^{L}\underset{i}{E}\Big[\frac{M_i}{m_i}\sum_{j=1}^{m_i}\underset{j}{E}^{(i)}\Big(\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}x_{ijk}'\Big)\Big]$$

$$+\sum_{i=1}^{L}\underset{i}{E}\Big[\frac{M_i}{m_i}\sum_{j=1}^{m_i}\underset{j}{E}^{(i)}\Big(\frac{N_{ij}}{n_{ij}}\sum_{k=1}^{k_{ij}}y_{ijk}'\Big)\Big]$$

$$=\sum_{i=1}^{L}\underset{i}{E}\Big(\frac{M_i}{m_i}\sum_{j=1}^{m_i}X_{ij}\Big)+\sum_{i=1}^{L}\underset{i}{E}\Big(\frac{M_i}{m_i}\sum_{j=1}^{m_i}Y_{ij}\Big)$$

$$=\sum_{i=1}^{L}X_i+\sum_{i=1}^{L}Y_i$$

$$=X+Y$$

$$=T$$

〈Proof (2)〉

$$\frac{1}{N_{ij}}\sum_{k=1}^{N_{ij}}\Big(y_{ijk}-\frac{Y_{ij}}{N_{ij}}\Big)^2=\frac{1}{N_{ij}}\Big\{\sum_{k=1}^{K_{ij}}\Big(y_{ijk}-\frac{Y_{ij}}{N_{ij}}\Big)^2+\sum_{k=K_{ij}+1}^{N_{ij}}\Big(y_{ijk}-\frac{Y_{ij}}{N_{ij}}\Big)^2\Big\}$$

$$=\frac{1}{N_{ij}}\Big\{\sum_{k=1}^{K_{ij}}\Big(y_{ijk}-\frac{Y_{ij}}{N_{ij}}\Big)^2+(N_{ij}-K_{ij})\Big(\frac{Y_{ij}}{N_{ij}}\Big)^2\Big\}$$

$$=\sigma_{2ij}^2$$

Since $V(\hat{T})\ \underset{j}{V_c}[E^{(i)}(\hat{T})]+\underset{i}{E}[\underset{j}{V}^{(i)}(\hat{T})$, therefore

$$\underset{j}{E}^{(i)}(\hat{T})=\underset{j}{E}^{(i)}\Big[\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}x_{ijk}'+\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{N_{ij}}{n_{ij}}\sum_{k=1}^{k_{ij}}y_{ijk}'\Big]$$

$$=\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\Big[\underset{j}{E}^{(i)}\Big(\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}x_{ijk}'+\frac{N_{ij}}{n_{ij}}\sum_{k=1}^{k_{ij}}y_{ijk}'\Big)\Big]$$

$$=\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}(X_{ij}+Y_{ij})$$

$$\underset{i}{V}\Big[\underset{j}{E}^{(i)}(\hat{T})\Big]=\underset{i}{V}\Big(\sum_{i=1}^{L}\sum_{j=1}^{m_i}(X_{ij}+Y_{ij})\Big)$$

$$=\sum_{i=1}^{L}\underset{i}{V}\Big(\frac{M_i}{m_i}\sum_{j=1}^{m_i}(X_{ij}+Y_{ij})\Big)$$

$$=\sum_{i=1}^{L}M_i^2\Big(\frac{M_i-m_i}{M_i-1}\Big)\Big(\frac{\sigma_{1i}^2+\sigma_{2i}^2}{m_i}\Big)$$

$$\underset{j}{V}^{(i)}(\hat{T})=\sum_{i=1}^{L}\underset{j}{V}^{(i)}\Big[\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}x_{ijk}'+\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{N_{ij}}{n_{ij}}\sum_{k=1}^{k_{ij}}y_{ijk}'\Big]$$

$$=\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)^2\sum_{j=1}^{m_i}\Big[M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\frac{\sigma_{1ij}^2}{m_{ij}}+N_{ij}^2\Big(\frac{N_{ij}-n_{ij}}{N_{ij}-1}\Big)\frac{\sigma_{2ij}^2}{n_{ij}}\Big]$$

$$E_i\Big(V_j^{(i)}(\hat{T})\Big)=E_i\Big[\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)^2\sum_{j=1}^{m_i}\Big\{M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\frac{\sigma_{1ij}^2}{m_{ij}}+N_{ij}^2\Big(\frac{N_{ij}-n_{ij}}{N_{ij}-1}\Big)\frac{\sigma_{2ij}^2}{n_{ij}}\Big\}\Big)$$

$$=\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)^2\Big(\frac{m_i}{M_i}\Big)\Big[\sum_{j=1}^{M_i}M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\frac{\sigma_{1ij}^2}{m_{ij}}+\sum_{j=1}^{M_i}N_{ij}^2\Big(\frac{N_{ij}-n_{ij}}{N_{ij}-1}\Big)\frac{\sigma_{2ij}^2}{n_{ij}}\Big)$$

$$V(\hat{T})=\sum_{i=1}^{L}\Big[M_i^2\Big(\frac{M_i-m_i}{M_i-1}\Big)\Big(\frac{\sigma_{1i}^2+\sigma_{2i}^2}{m_i}\Big)+\frac{M_i}{m_i}\sum_{j=1}^{M_i}M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\frac{\sigma_{1ij}^2}{m_{ij}}$$

$$+\frac{M_i}{m_i}\sum_{j=1}^{M_i}N_{ij}^2\Big(\frac{N_{ij}-n_{ij}}{N_{ij}-1}\Big)\frac{\sigma_{2ij}^2}{n_{ij}}\Big]$$

<Proof (3)> $\frac{1}{n_{ij}-1}\sum_{k=1}^{n_{ij}}\Big(y_{ijk}'-\frac{y_{ij}'}{n_{ij}}\Big)^2$

$$=\frac{1}{n_{ij}-1}\Big[\sum_{k=1}^{k_{ij}}\Big(y_{ijk}'-\frac{y_{ij}'}{n_{ij}}\Big)^2+\sum_{k=k_{ij}+1}^{n_{ij}}\Big(y_{ijk}'-\frac{y_{ij}'}{n_{ij}}\Big)^2\Big]$$

$$=\frac{1}{n_{ij}-1}\Big[\sum_{k=1}^{k_{ij}}\Big(y_{ijk}'-\frac{y_{ij}'}{n_{ij}}\Big)^2+(n_{ij}-k_{ij})\frac{(y_{ij}')^2}{(n_{ij})^2}\Big]$$

$$=S_{2ij}^2$$

$E(S_{1i}^2)=\sigma_{1i}^2,\ E(S_{2i}^2)=\sigma_{2i}^2, E(S_{1ij}^2)=\sigma_{1ij}^2,$

$E(S_{2ij}^2)=\sigma^2_{2ij},$ so that,

$$E[\hat{V}(\hat{T})]=V(\hat{T}).$$

Theorem4-2: In the q-th sampling in the sequential procedures, the variance of the estimated total $\hat{T}$ in Theorem 4-1, is a minimum if m_{ij} and n_{ij} are proportional to $\frac{\sqrt{M_i}M_{ij}\sigma_{1ij}}{\sqrt{C_{1ij}m_i}}$ and $\frac{\sqrt{M_i}N_{ij}\sigma_{2ij}}{\sqrt{C_{2ij}m_i}}$, respectively, where C_{1ij} and C_{2ij} are cost per unit sampling from A_q and D_{q-1} in ij-th cluster.

<Proof> Let C_0 represents an overhead cost.

Then the total cost is

$$C=C_0+\sum_{i=1}^{L}\sum_{j=1}^{M_i}(C_{1ij}m_{ij}+C_{2ij}n_{ij})$$

The problem is to minimize $V(\hat{T})$ subject to the restriction of C. We use the method of Lagrangerian multipriers and select the m_{ij} and n_{ij} to minimize

$$f(m_{ij}, n_{ij})=V(\hat{T})+\lambda\Big[C_0+\sum_{i=1}^{L}\sum_{j=1}^{M_i}(C_{1ij}m_{ij}+C_{2ij}n_{ij})\Big].$$

That is,

$$\frac{\partial f}{\partial m_{ij}}=\sum_{i=1}^{L}\sum_{j=1}^{M_i}\Big(\frac{M_i}{m_i}\Big)\frac{M_{ij}^2(-\sigma_{1ij}^2)}{m_{ij}^2}+\lambda\sum_{i=1}^{L}\sum_{j=1}^{M_i}C_{1ij}=0$$

$$\frac{\partial f}{\partial n_{ij}}=0$$

As the results,

$$m_{ij}=\frac{\sqrt{M_i}M_{ij}\sigma_{1ij}}{\sqrt{\lambda C_{1ij}m_i}},\quad n_{ij}=\frac{\sqrt{M_i}N_{ij}\sigma_{2ij}}{\sqrt{\lambda C_{2ij}m_i}}.$$

If the total size of sampling is given, say n', then we get.

$$m_{ij}=\frac{n'M_{ij}\sigma_{1ij}}{\sqrt{C_{1ij}}\left(\frac{M_{ij}\sigma_{1ij}}{\sqrt{C_{1ij}}}+\frac{N_{ij}\sigma_{2ij}}{\sqrt{C_{2ij}}}\right)},\quad n_{ij}=\frac{n'N_{ij}\sigma_{2ij}}{\sqrt{C_{2ij}}\left(\frac{M_{ij}\sigma_{1ij}}{\sqrt{C_{1ij}}}+\frac{N_{ij}\sigma_{2ij}}{\sqrt{C_{2ij}}}\right)}$$

5. Three-stage Stratified Sampling in Sequential Procedures.

The population is divided into nonoverlapping subpopulations and i-th stratum has M_i primary sampling units (P. S. U.), E_{i1}, E_{i2}, ...$E_{iM_i}(i=1,......L)$.

E_{ij} isagain divided into M_{ij} nonoverlapping which are secondary sampling units (S. S. U.), $E_{ij1}, E_{ij2}, ...E_{ij}M_{ij}(i=1,......L, j=1,......M_i)$.

$E_{ijk}(i=1,......L, j=1,......M_i, \ k=1,......M_{ij})$ is also divided into nonoverlapping clusters which are consisted with A_q, B_q and C_q. Then $\theta_{ijkl}\epsilon A_q(l=1,......M_{ijk})$, $w_{ijkl}\epsilon B_q$ $(l=1,......K_{ijk})$, $w_{ijkl}\epsilon\ C_q(l=K_{ijk}+1,......N_{ijk})$.

Therorem5-1: Let the value of a character α in the clusters be

$$x_{ijkl}=x(\theta_{ijkl}),\quad (l=1,......M_{ijk})$$
$$y_{ijkl}=y(w_{ijkl}),\quad (l=1,......K_{ijk})$$
$$y_{ijkl}=y(w_{ijkl})=0,\quad (l=K_{ijk}+1,......N_{ijk}),\ \text{then}$$

in the q-th sampling of sequential procedures,

(1) $$\hat{T}=\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}\frac{M_{ijk}}{m_{ijk}}\sum_{l=1}^{m_{ij}}x_{ijkl}'+\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}\frac{N_{ijk}}{n_{ijk}}\sum_{l=1}^{k_{ij}}y_{ijkl}'$$

is unbiased estimate of the total $T=\sum_{i=1}^{L}\sum_{j=1}^{M_i}\sum_{k=1}^{M_{ij}}\left(\sum_{l=1}^{M_{ijk}}x_{ijkl}+\sum_{l=1}^{K_{ijk}}y_{ijkl}\right)$ of the true population π_q.

(2) The variance of $\hat{T}$ is

$$V(\hat{T})=\sum_{h=1}^{L}\Big[M_i^2\left(\frac{M_i-m_i}{M_i-1}\right)\left(\frac{\sigma_{1i}^2+\delta_{2i}^2}{m_i}\right)+\frac{M_i}{m_i}\sum_{j=1}^{M_i}M_{ij}^2\left(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\right)\left(\frac{\sigma_{1ij}^2+\sigma_{2ij}^2}{m_{ij}}\right)$$
$$+\frac{M_i}{m_i}\sum_{j=1}^{M_i}\left\{\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{M_{ij}}M_{ijk}^2\left(\frac{M_{ijk}-m_{ijk}}{M_{ijk}-1}\right)\frac{\sigma_{1ijk}^2}{m_{ijk}}\right\}$$
$$+\frac{M_i}{m_i}\sum_{j=1}^{M_i}\left\{\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{M_{ij}}N_{ijk}^2\left(\frac{N_{ijk}-n_{ijk}}{N_{ijk}-1}\right)\frac{\sigma_{2ijk}^2}{n_{ijk}}\right\}\Big)$$

where

$$\sigma_{1ijk}^2=\frac{1}{M_{ijk}}\sum_{l=1}^{M_{ijk}}(x_{ijkl}-\bar{x}_{ijk})^2,\quad \bar{x}_{ijk}=\frac{X_{ijk}}{M_{ijk}}$$

$$\sigma^2_{2ijk}=\frac{1}{N_{ijk}}\sum_{l=1}^{K_{ijk}}(y_{ijkl}-\bar{y}_{ijk})^2,\quad \bar{y}_{ijk}=\frac{Y_{ijk}}{N_{ijk}}$$

$$X_{ijk}=\sum_{l=1}^{M_{ijk}}x_{ijkl},\quad Y_{ijk}=\sum_{l=1}^{N_{ijk}}y_{ijkl}$$

(3) unbiased estimate of $V(\hat{T})$ is

$$\hat{V}(\hat{T})=\sum_{i=1}^{L}\Big[M_i^2\Big(\frac{M_i-m_i}{M_i-1}\Big)\Big(\frac{S_{1i}^2+S_{2i}^2}{m_i}\Big)+\frac{M_i}{m_i}\sum_{j=1}^{M_i}M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\Big(\frac{S_{1ij}^2+S_{2ij}^2}{m_{ij}}\Big)$$
$$+\frac{M_i}{m_i}\sum_{j=1}^{M_i}\Big\{\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{M_{ij}}M_{ijk}^2\Big(\frac{M_{ijk}-m_{ijk}}{M_{ijk}-1}\Big)\frac{S_{1ijk}^2}{m_{ijk}}\Big\}$$
$$+\frac{M_i}{m_i}\sum_{j=1}^{M_i}\Big\{\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{M_{ij}}N_{ijk}^2\Big(\frac{N_{ijk}-n_{ijk}}{N_{ijk}-1}\Big)\frac{S_{2ijk}^2}{n_{ijk}}\Big\}\Big]$$

where,

$$S_{1ijk}^2=\frac{1}{m_{ijk}-1}\sum_{l=1}^{m_{ijk}}(x_{ijkl}'-\bar{x}_{ijk}')^2$$

$$S_{2ijk}^2=\frac{1}{n_{ijk}-1}\Big[\sum_{l=1}^{k_{ijk}}(y_{ijkl}'-\bar{y}_{ijk}')^2+(n_{ijk}-k_{ijk})(\bar{y}_{ijk}')^2\Big]$$

$$\bar{x}_{ijk}'=\frac{x'_{ijk}}{m_{ijk}},\quad \bar{y}_{ijk}'=\frac{y_{ijk}'}{n_{ijk}},\quad x_{ijk}'=\sum_{l=1}^{m_{ijk}}x_{ijkl}',$$

$$y_{ijk}'=\sum_{l=1}^{k_{ijk}}y_{ijkl}.$$

〈Proof (1)〉 Since $E(\hat{T})=\underset{i}{E}\underset{j}{E^{(i)}}\underset{k}{E^{(ij)}}(\hat{T})$,

$$\underset{i}{E}\underset{j}{E^{(i)}}\underset{k}{E^{(ij)}}\Big[\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}\frac{M_{ijk}}{m_{ijk}}\sum_{l=1}^{m_{ijk}}x_{ijkl}'$$
$$+\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}\frac{N_{ijk}}{n_{ijk}}\sum_{l=1}^{k_{ijk}}y_{ijkl}'\Big]$$
$$=\underset{i}{E}\underset{j}{E^{(i)}}\Big[\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}\underset{k}{E^{(ij)}}\Big\{\frac{M_{ijk}}{m_{ijk}}\sum_{l=1}^{m_{ijk}}x_{ijkl}'+\frac{N_{ijk}}{n_{ijk}}\sum_{l=1}^{k_{ijk}}y_{ijkl}'\Big\}\Big]$$
$$=\underset{i}{E}\underset{j}{E^{(i)}}\Big[\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}(X_{ijk}+Y_{ijk})\Big]$$
$$=\underset{i}{E}\Big[\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\underset{j}{E^{(i)}}\Big\{\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}(X_{ijk}+Y_{ijk})\Big\}\Big]$$
$$=\underset{i}{E}\Big[\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}(X_{ij}+Y_{ij})\Big]$$
$$=\sum_{i=1}^{L}(X_i+Y_i)$$
$$=T$$

〈Proof (2)〉

$$V_{ijk}(\hat{T})=V_i\Big[E_j^{(i)}(E_k^{(ij)}(\hat{T}))\Big]+E_i\Big[V_j^{(i)}(E_k^{(ij)}(\hat{T}))\Big]+E_i\Big[E_j^{(i)}(V_k^{(ij)}(\hat{T}))\Big]$$

$$E_k^{(ij)}(\hat{T})=\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{m_{ij}}(X_{ijk}+Y_{ijk})$$

$$E_j^{(i)}\Big[E_k^{(ij)}(\hat{T})\Big]=\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{m_i}(X_{ij}+Y_{ij})$$

$$V_i\Big[E_j^{(i)}(E_k^{(ij)}(\hat{T}))\Big]=\sum_{i=1}^{L}M_i^2\Big(\frac{M_i-m_i}{M_i-1}\Big)\Big(\frac{\sigma_{1i}^2+\sigma_{2i}^2}{m_i}\Big)$$

$$V_j^{(i)}(E_k^{(ij)}(\hat{T}))\Big]=\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)^2\sum_{j=1}^{m_i}M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\Big(\frac{\sigma_{1ij}^2+\sigma_{2ij}^2}{m_{ij}}\Big)$$

$$E_i\Big[V_j^{(i)}(E_k^{(ij)}(\hat{T}))\Big]=\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)\sum_{j=1}^{M_i}M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\Big(\frac{\sigma_{1ij}^2+\sigma_{2ij}^2}{m_{ij}}\Big)$$

$$V_k^{(ij)}(\hat{T})=\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)^2\sum_{j=1}^{m_i}\Big(\frac{M_{ij}}{m_{ij}}\Big)^2\sum_{k=1}^{m_{ij}}\Big[M_{ijk}^2\Big(\frac{M_{ijk}-m_{ijk}}{M_{ijk}-1}\Big)\frac{\sigma_{1ijk}^2}{m_{ijk}}$$
$$+N_{ijk}^2\Big(\frac{N_{ijk}-n_{ijk}}{N_{ijk}-1}\Big)\frac{\sigma_{2ijk}^2}{n_{ijk}}\Big]$$

$$E_j^{(i)}\Big[V_k^{(ij)}(\hat{T})\Big]=\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)\sum_{j=1}^{m_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{M_{ij}}\Big\{M_{ijk}^2\Big(\frac{M_{ijk}-m_{ijk}}{M_{ijk}-1}\Big)\frac{\sigma_{1ijk}^2}{m_{ijk}}$$
$$+N_{ijk}^2\Big(\frac{N_{ijk}-n_{ijk}}{N_{ijk}-1}\Big)\frac{\sigma_{2ijk}^2}{n_{ijk}}\Big\}$$

$$E_i\Big[E_j^{(i)}V_k^{(ij)}(\hat{T})\Big]=\sum_{i=1}^{L}\frac{M_i}{m_i}\sum_{j=1}^{M_i}\frac{M_{ij}}{m_{ij}}\sum_{k=1}^{M_{ij}}\Big[M_{ijk}^2\Big(\frac{M_{ijk}-m_{ijk}}{M_{ijk}-1}\Big)\frac{\sigma_{1ijk}^2}{m_{ijk}}$$
$$+N_{ijk}^2\Big(\frac{N_{ijk}-n_{ijk}}{N_{ijk}-1}\Big)\frac{\sigma_{2ijk}^2}{n_{ijk}}\Big]$$

Therefore

$$V(\hat{T})=\sum_{i=1}^{L}M_i^2\Big(\frac{M_i-m_i}{M_i-1}\Big)\Big(\frac{\sigma_{1i}^2+\sigma_{2i}^2}{m_i}\Big)+\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)\sum_{j=1}^{M_i}M_{ij}^2\Big(\frac{M_{ij}-m_{ij}}{M_{ij}-1}\Big)\Big(\frac{\sigma_{1ij}^2+\sigma_{2ij}^2}{m_{ij}}\Big)$$
$$+\sum_{i=1}^{L}\Big(\frac{M_i}{m_i}\Big)\sum_{j=1}^{M_i}\Big(\frac{M_{ij}}{m_{ij}}\Big)\sum_{k=1}^{M_{ij}}\Big[M_{ijk}^2\Big(\frac{M_{ijk}-m_{ijk}}{M_{ijk}-1}\Big)\frac{\sigma_{1ijk}^2}{m_{ijk}}$$
$$+N_{ijk}^2\Big(\frac{N_{ijk}-n_{ijk}}{N_{ijk}-1}\Big)\frac{\sigma_{2ijk}^2}{n_{ijk}}\Big]$$

⟨Proof (3)⟩

Since

$$S_{2ijk}^2=\frac{1}{n_{ijk}-1}\sum_{l=1}^{n_{ijk}}(y_{ijkl}'-\bar{y}_{ijk}')^2$$
$$=\frac{1}{n_{ijk}-1}\Big[\sum_{l=1}^{k_{ijk}}\Big(y_{ijkl}'-\frac{y_{ijk}'}{n_{ijk}}\Big)^2+\sum_{l=k_{ijk}+1}^{n_{ijk}}\Big(y_{ijkl}'-\frac{y_{ijk}'}{n_{ijk}}\Big)^2\Big]$$
$$=\frac{1}{n_{ijk}-1}\Big[\sum_{l=1}^{k_{ijk}}\Big(y_{ijkl}'-\frac{y_{ijk}'}{n_{ijk}}\Big)^2+(n_{ijk}-k_{ijk})\Big(\frac{y_{ijk}'}{n_{ijk}}\Big)^2\Big]$$

and $E(S_{1i}^2)=\sigma_{1i}^2$, $E(S_{2i}^2)=\sigma_{2i}^2$, $E(S_{1ij}^2)=\sigma_{1ij}^2$, $E(S_{2ij}^2)=\sigma_{2ij}^2$, $E(S_{1ijk}^2)=\sigma_{1ijk}^2$, $E(S_{2ijk}^2)$

$=\sigma_{2ijk}{}^2$

so that

$$E\hat{V}(\hat{T})=V(\hat{T})$$

Theorem 5-2: In the q-th sampling of sequential procedures, the variance of the estimated total $\hat{T}$ in Theorem 5-1 is the minimum when m_{ijk} and n_{ijk} are proportional to

$$\frac{\sqrt{M_iM_{ij}}\,M_{ijk}\sigma_{1ijk}}{\sqrt{C_{1ijk}m_im_{ij}}} \text{ and } \frac{\sqrt{M_iM_{ij}}\,N_{ijk}\sigma_{2ijk}}{\sqrt{C_{2ijk}m_im_{ij}}},$$

respectively, where C_{1ijk} and C_{2ijk} are the cost per unit sampling from A_q and D_{q-1} in E_{ijk} clusters.

〈Proof〉 Similar to Theorem 4-2, we can prove Theorem 5-2, and if total size of sampling is given, say $n'=m_{ijk}+n_{ijk}$ then we can get

$$m_{ijk}=\frac{n'M_{ijk}\,\sigma_{1ijk}}{\sqrt{C_{1ijk}}\left(\dfrac{M_{ijk}\,\sigma_{1ijk}}{\sqrt{C_{1ijk}}}+\dfrac{N_{ijk}\,\sigma_{2ijk}}{\sqrt{C_{2ijk}}}\right)},$$

$$n_{ijk}=\frac{n'N_{ijk}\,\sigma_{2ijk}}{\sqrt{C_{2ijk}}\left(\dfrac{M_{ijk}\,\sigma_{1ijk}}{\sqrt{C_{2ijk}}}+\dfrac{N_{ijk}\,\sigma_{2ijk}}{\sqrt{C_{2ijk}}}\right)}$$

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