INVITED LECTURE

Nonlinear Models in Analysis of Variance and a Functional Equation

Seiji Nabeya*

Part I. Nonlinear Models in Analysis of Variance (Abstract from Doksum [1])

In the one-way layout the linear model in the analysis of variance can be written in the form

$$Z_{ij}=\mu_i+E_{ij},\ \ j=1,\ldots\ldots,n_i;\ \ i=1,\ldots\ldots,k$$

where Z_{ij} are the observable random variables, $\mu_1,\ldots\ldots,\mu_k$ are constants to be compared, and E_{ij} are i. i. d. (independently and identically distributed).

We consider in particular the two sample case. Let $X_1,\ldots\ldots,X_m$ be i. i. d. according to F and $Y_1,\ldots\ldots,Y_n$ be i. i. d. according to G. In the case of the linear model there exists a constant Δ, such that $F(x)=G(x+\Delta)$ for all x.

When the linear model assumption is not satisfied, we consider the shift function

$$\Delta(x)=\inf\{\Delta;\ F(x)\leqq G(x+\Delta)\} \quad (1)$$
$$=G^{-1}(F(x))-x$$

introduced by Lehmann [3], where G^{-1} is defined by

$$G^{-1}(u)=\inf\{x;u\leqq G(x)\}.$$

$\Delta(x)$ is defined for all points of support $S(F)$ of F:

$$x\in S(F)=\{x;\ 0<F(x)<1\}.$$

Theorem 1.1 For arbitrary distribution functions F and G, $x+\Delta(x)$ is non-decreasing and $X+\Delta(X)$ is stochastically no smaller than Y.

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*Profcsser of Statistics, Faculty of Economics, Hitotsbashi University, Tokyo, Japan. This article will be presented at the 1973 Annual Meeting of the Korean Statistical Society to be held on Dec. 22, 1973, Seoul, Korea.

Theorem 1.2 (i) If F is continuous, then $X+\Delta(X)$ has the same distribution as Y.
(ii) If F is continuous, if $X+\Delta^*(x)$ has the same distribution as Y, and if $x+\Delta^*(x)$ is non-decreasing, then $\Delta^*(x)=\Delta(x)$ a.s. in $S(F)$.

In a similar way as (1) we define

$$\theta(y)=\sup\{\theta;\ G(y)\leqq F(y-\theta)\}$$
$$=y-F^{-1}(G(y))$$

for $y\in S(G)$.

Theorem 1.3 Suppose that F and G are continuous.

(i) If F is strictly increasing on $S(F)$, then

$$\Delta(x)=\theta(x+\Delta(x)) \qquad \text{for } x\in S(F).$$

(ii) If G is strictly increasing on $S(G)$, then

$$\theta(x)=\Delta(x-\theta(x)) \qquad \text{for } x\in S(G).$$

Theorem 1.4 Let Δ be a constant. If F is strictly increasing on $S(F)$ and if $F(x)=G(x+\Delta)$ for all x, then $\Delta(x)=\Delta$ for $x\in S(F)$ and $\theta(x)=\Delta$ for $x\in S(G)$.

Theorem 1.5 Let F and G be continuous and strictly increasing. If $\Delta(x)=\theta(x)$ for all $x\in R$, then there is a constant Δ such that $F(x)=G(x+\Delta)$ for all $x\in R$.

<Proof> The equation $\Delta(x)=\theta(x)$ can be written as

(2) $$G^{-1}(F(x))-x=x-F^{-1}(G(x)).$$

If we put $f(x)=G^{-1}(F(x))$, then f is also continuous and strictly increasing, and (2) reduces to

(3) $$f(x)=2x-f^{-1}(x).$$

Inserting $f(x)$ in place of x in (3) we have

(4) $$f(f(x))=2f(x)-x.$$

In Theorem 2.3 below we can state that any continuous function f satisfying (4) must be of the form $f(x)=x+\Delta$ for some constant Δ, *from which it follows that* $G^{-1}(F(x))=x+\Delta$, hence $F(x)=G(x+\Delta)$.

Note: If we put $g(x)=f(x)-x$, then (4) is converted to

(5) $$g(x+g(x))=g(x).$$

This is called Euler's equation and it was solved by Wagner [5] for the first time.

Let F_m and G_n be empirical distribution functions of $X_1, \ldots\ldots, X_m$ and $Y_1, \ldots\ldots, Y_n$, respectively. Then

$$\hat{\Delta}(x)=G_n^{-1}(F_m(x))-x$$

is clearly a nonparametric estimate of $\Delta(x)$.

On the contrary, if F is $N(\mu_1, \sigma_1^2)$ and G is $N(\mu_2, \sigma_2^2)$, then we have

$$\Delta(x) = \frac{\sigma_2}{\sigma_1}(x-\mu_1) + \mu_2 - x,$$

hence

$$\hat{\Delta}_0(x) = \frac{S_2}{S_1}(x-\bar{X}) + \bar{Y} - x$$

is a parametric estimate of $\Delta(x)$.

Doksum develops a theory of confidence bands for $\Delta(x)$ and he also obtains the asymptotic distribution of the stochastic process $\sqrt{m+n}\,(\hat{\Delta}(x) - \Delta(x))$ under suitable conditions.

Part II. Functional Equation $f(p+qx+cf(x)) = a+bx+cf(x)$ (Abstract from Nabeya[4])

I succeeded in finding all the continuous solutions of the functional equation

$$f(p+qx+rf(x)) = a+bx+cf(x), \tag{6}$$

for any given constants $p, q, r, a, b,$ and c. This is a generalized form of (4) and (5).

The functional equation (6) with $r=0$ is a special case of the equations treated in Kuczma [2], and is not so interesting in our case. If $r \neq 0$, then (6) can be converted to

$$g(g(x)) = p - cp + ar + (br - cq)x + (c+q)g(x)$$

by putting

$$g(x) = p + qx + rf(x).$$

Hence it is sufficient to consider the functional equation

$$f(f(x)) = a + bx + cf(x). \tag{7}$$

We assume further $b \neq 0$ in (7), because $b=0$ leads to a simpler equation

$$f(f(x)) = a + cf(x),$$

which can be easily solved.

Theorem 2.1 Let f be a continuous solution of (7) with $b \neq 0$. Then f is strictly monotone and it maps R continuously onto itself.

By this theorem we can define the continuous and strictly monotone inverse function f^{-1}. If we define furthermore

$$f^0(x) = x,\ f^{n+1}(x) = f^n(f(x)),\ f^{-n-1}(x) = f^{-n}(f^{-1}(x)),\ n=1, 2, \ldots\ldots,$$

then $f^n(x)$ $(n = \ldots\ldots, -1, 0, 1, \ldots\ldots)$ can be written as

$$f^n(x) = a_n + b_n x + c_n f(x),$$

where a_n, b_n and c_n satisfy a system of linear homogeneous difference equations:

(8) $$a_{n+1}=a_n+ac_n,\quad b_{n+1}=bc_n,\quad c_{n+1}=b_n+cc_n.$$

The characteristic equation of (8) is

$$(\rho-1)(\rho^2-c\rho-b)=0,$$

which always has a root $\rho=1$. We denote the other two roots by ρ_1 and ρ_2. The nature of the solutions of (7) differs very much depending on the values of ρ_1 and ρ_2.

I shall state the results for two cases.

Theorem 2.2 Let $1<\rho_1<\rho_2$.

(i) Then we have two linear solutions

$$\phi_1(x)=\rho_1 x-\frac{a}{\rho_2-1} \text{ and } \phi_2(x)=\rho_2 x-\frac{a}{\rho_1-1},$$

which intersect at $\xi=a/(1-b-c)$.

(ii) Every continuous solution f of (7) satisfies

$$\phi_1(x)\geqq f(x)\geqq\phi_2(x) \text{ if } x<\xi,$$
$$\phi_1(x)\leqq f(x)\leqq\phi_2(x) \text{ if } x>\xi,$$
$$f(\xi)=\xi,$$

and

(9) $$\rho_1\leqq\frac{f(x)-f(y)}{x-y}\leqq\rho_2 \quad \text{for any } x\neq y.$$

(iii) Let x_0 and x_1 be any two values such that

$$x_0>\xi \text{ and } \phi_1(x_0)\leqq x_1\leqq\phi_2(x_0)$$

and define $x_2=a+bx_0+cx_1$. Let f_1 be any continuous function defined in the interval $[x_0, x_1]$ satisfying $f_1(x_0)=x_1$, $f_1(x_1)=x_2$ and the condition (9) in the interval $[x_0, x_1]$. Then we can construct a continuous solution f of (7) in the interval (ξ, ∞), which coincides with f_1 in the interval $[x_0, x_1]$.

Similar situation holds also for the interval $(-\infty, \xi)$.

(iv) Any continuous solution of (7) is differentiable to the right and to the left at $x=\xi$.

Theorem 2.3 Let $\rho_1=\rho_2=1$.

(i) If $a\neq 0$, then (7) does not have a continuous solution.

(ii) If $a=0$, then every continuous solution f of (7) is of the form

(10) $$f(x)=x+\Delta$$

for some constant Δ. Any function of the form (10) is a solution of (7).

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