〈韓國數學敎育學會誌數學敎育〉
1973. 9. Vol. XII No. 1

Journal of the Korean Society of Mathematical Education
Sept. 1973. Vol. XII No. 1

A semi-exact in tensor product

by

Chul-Kon Bae, Im-Suk Lee and Kang-Joo Min

[1] **ABSTRACT.** In this paper, we want to verify some properties in tensor product. It is interesting to think semi-exact sequence in tensor product by [3]. Moreover no hardness is there in process and we want to discuss the commutativity in tensor product. For a certain semi-exact sequence, if we product arbitrary Abelian group for each group then the tensor product will do or not. Here, we have positive answer. At first we define the semi-exact sequence as following.

DEFINITION. A finite or infinite sequence

$$\cdots\cdots \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \cdots\cdots$$

of homomorphisms of Abelian groups is said to be semi-exact iff gof is trivial homomorphism. Ocassionary one says it is semi-exact iff the image of input homomorphism is contained in the kernel of the output homomorphism at every group over than the ends (if any) of the sequence.

DEFINITION. (T,f) is a tensor product of the Abelian groups A and B iff an Abelian group T together with a bi-additive function

$$f: A\times B \longrightarrow T$$

such that, for every bi-additive function

$$g: A\times B \longrightarrow X$$

from $A\times B$ into an Abelian group X, there exists a unique homororphism $h: T\longrightarrow X$ which satisfies the commutative relation

$$h\circ f = g$$

in the following triangle:

$$\begin{array}{ccc} A\times B & \xrightarrow{f} & T \\ & g\searrow \quad \swarrow h & \\ & X & \end{array}$$

Thus T will be denoted by the symbol

$$A\otimes B$$

and bi-additive function f will be denoted by the symbol

$$\tau: A\times B \longrightarrow A\otimes B$$

and called the tensor map.

For each $a\in A$ and $b\in B$, the element of $A\otimes B$ will be denoted by

$$a\otimes b$$

and called the tensor product of the elements a and b.

In [1], every element of $A\otimes B$ can be written in the form

$$t=\sum_{i=1}^{n}(a_i\otimes b_i)$$

where $a_i \in A$ and $b_i \in B$ for every $i=1, 2, \cdots\cdots, n$.

The notations and concepts are based on [2].

[II] **TENSOR PRODUCT.** In this section, we will denote $T=A\otimes B$ tensor product of Abelian groups A and B.

LEMMA1. For arbitrary $t\in T$ $(t=\sum_{k=1}^{n}(a_k\otimes g_k))$, $f\otimes i\sum_{k=1}^{n}(a_k\otimes g_k)=\sum_{k=1}^{n}(f(a_k)\otimes i(g_k))$

PROOF.

$$
\begin{aligned}
f\otimes i\sum_{k=1}^{n}(a_k\otimes g_k) &= (f\otimes i)\sum_{k=1}^{n}(\tau(a_k, g_k)) \\
&= (f\otimes i)<\tau(a_1, g_1)+\cdots\cdots+\tau(a_n, g_n)\} \\
&= (f\otimes i)\tau(a_1, g_1)+\cdots\cdots+(f\otimes i)\tau(a_n, g_n) \\
&= (f(a_1)\otimes i(g_1))+\cdots\cdots+(f(a_n)\otimes i(g_n) \\
&= \sum_{k=1}^{n}(f(a_k)\otimes i(g_k)).
\end{aligned}
$$

THEOREM2. A finite or infinite semi-exact sequence

$$\cdots\cdots\longrightarrow A\xrightarrow{f}B\xrightarrow{g}C\longrightarrow\cdots\cdots$$

Then

$$\cdots\cdots\longrightarrow A\otimes G\xrightarrow{f_*}B\otimes G\xrightarrow{g_*}C\otimes G\longrightarrow\cdots\cdots$$

is semi-exact if G be an arbitrary Abelian group and i is trivial endomorphism.

PROOF. We prove, for any two consecutive homomorphisms f_* and g_*, $g_*\circ f_*$ is trivial homomorphism. We define f_* and g_* as follows;

$$f_*=f\otimes j \wedge g_*=g\otimes j$$

We clam $g_*\circ f_*$ is trivial homomorphism. For arbitrary $t\in A\otimes G$, let $t=\sum_{k=1}^{n}(a_k\otimes g_k)$ for $a_k\in A$ and $g_k\in G$ for each $k=1, 2, \cdots\cdots, n$. Then we have by above lemma

$$
\begin{aligned}
g_*\circ f_*(t) &= g_*(f\otimes j)\left(\sum_{k=1}^{n}(a_k\otimes g_k)\right) \\
&= g_*\left(\sum_{k=1}^{n}(f(a_k)\otimes j(g_k)\right)
\end{aligned}
$$

And we apply the same method and lemma

$$
\begin{aligned}
g_*\circ f_*(t) &= \sum_{k=1}^{n}(g\circ f)(a_k)\otimes(j\circ j)(g_k) \\
&= e_A\otimes e_G.
\end{aligned}
$$

It means the unit element of $A\otimes G$. This completes the proof.

REMARK 3. In fact, in above theorem, we must say for the commutativity. It is clear because of that if t and t′ be the elements of $A\otimes G$, then we can put $t=\sum_{i=1}^{m}(a_i\otimes g_i)$ for $a_i\in A$ and $g_i\in G$ and $t'=\sum_{j=1}^{n}(a_j\otimes g_j)$ for $a_j\in A$ and $g_j\in G$. So

$$
\begin{aligned}
t\cdot t' &= \sum_{i=1}^{m}(a_i\otimes g_i)\sum_{j=1}^{n}(a_j\otimes g_j) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}(a_ia_j\otimes g_ig_j)=\sum_{j=1}^{n}\sum_{i=1}^{m}(a_ja_i\otimes g_jg_i)
\end{aligned}
$$

$a_i a_j = a_j a_i$ and $g_i g_j = g_j g_i$ by A is commutative and G is commutative

$$\sum_{j=1}^{n}\sum_{i=1}^{m}(a_j a_i \otimes g_j g_i) = \sum_{j=1}^{n}(a_j \otimes g_j)\sum_{i=1}^{m}(a_i \otimes g_i) = t' \circ t$$

THEOREM 4. In the following diagram

$$\begin{array}{ccccc} A\times G & \xrightarrow{f\times i} & B\times G & \xrightarrow{g\times j} & C\times G \\ \downarrow \tau_A & & \downarrow \tau_B & & \downarrow \tau_C \\ A\otimes G & \xrightarrow{f\otimes i} & B\otimes G & \xrightarrow{g\otimes j} & C\otimes G \end{array}$$

where the rows are semi-exact and the squares are commutative. Then

$$I_m(\tau_A) \xrightarrow{f^*} I_m(\tau_B) \xrightarrow{g^*} I_m(\tau_C)$$

is semi-exact.

PROOF. Tensor product of homomorphisms f^* and g^* is defined by

$$f^* = f\otimes i \,|\, I_m(\tau_A)$$
$$g^* = g\otimes j \,|\, I_m(\tau_B)$$

respectively. We claim the proposition that $g^* \circ f^*$ is trivial homomorphism, and by virture of the theorem 2, we can complete the proof.

References

[1] Sze-Tsen Hu, 1965. Elements of Modern Algebra.

[2] Jacob K. Goldhaber and Gertude Ehrlich, 1970. Algebra.

[3] Michel Barr, 1972, The existence of free groups, (Vol. 79, No4. A.M.M.)

[4] Chul-Kon Bae, 1972. A note on the semi-exact sequence.