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A PROPERTY OF NORMAL MEROMORPHIC FUNCTIONS WITH NON-NORMAL SUMS

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It is known ((1) and (2)) that the sums and the products of normal functions need not be normal. In this note we shall present a simple result concerning two meromorphic functions whose sum is not norma

If $f(z)$ and $g(z)$ are normal meromorphic functions in the unit disc D and $h(z)=f(z)+g(z)$ is non-normal in D, then there exists a sequence of points $\{z_n\}$ in D, $|z_n|\to 1$, such that, either

$$\lim(1-|z_n|)\ |f'(z_n)|=\infty \quad \text{or} \quad \lim(1-|z_n|)|g'(z_n)|=\infty$$

Assume the contrary. Then there exist positive constants $r_1(<1)$, $r_2(<1)$, K_1, and K_2 such that

$$|f'(z)|<\frac{K_1}{1-|z|} \text{ for every } z,\ r_1<|z|<1, \text{ and}$$

$$|g'(z)|<\frac{K_2}{1-|z|} \text{ for every } z,\ r_2<|z|<1.$$

Let $K=K_1+K_2$, $r=\max(r_1, r_2)$. Then

$$|h'(z)|=|f'(z)+g'(z)|<\frac{K}{1-|z|} \text{ for every } z, r<|z|<1.$$

We shall complete the proof by showing that $h(z)$ is normal.
We can choose r so that $f(z)$ has no pole or zero on $|z|=r$.
For z, $r<|z|<1$, we have

$$\frac{|h'(z)|}{1+|h(z)|^2}\leq|h'(z)|<\frac{K}{1-|z|}<\frac{2K}{1-|z|^2}$$

Let $|z|\leq r$. Since $h(z)$ has only finitely many poles in $|z|\leq r$, say at $z_1, z_2, \cdots\cdots, z_m$, then for a sufficiently small neighborhood of each z_i, $i=1,2,\cdots\cdots,m$, we have the following expansion for $h(z)$:

$$h(z)=a_{-n_i}(z-z_i)^{-n_i}+\cdots\cdots\cdots, \text{ where } a_{-n_i}\neq 0.$$

Thus $h'(z)=-n_i\ a_{-n_i}(z\text{-}z_i)_{-n_i-1}+\cdots\cdots$, in otherwords,

$$\left|\frac{h'(z)}{h(z)}\right|=\left|\frac{-n_i}{a_{-n_i}}(z-z_i)\right|^{2n_i-(n_i+1)}(1+0(z-z_i)$$

Hence, there exists a small positive real number d such that

$$\left|\frac{h'(z)}{h(z)}\right| < \left|\frac{n_i}{a_{-n_i}}\right| + 1 \quad \text{for } z,\ |z-z_i|<d.$$

Thus $\dfrac{|h'(z)|}{1+|h(z)|^2} < \left|\dfrac{n_i}{a_{-n_i}}\right| + 1$ for z, $|z-z_i|<d$

for some $i=1, 2, \cdots\cdots m$.

If z is in $|z|\leq r$ and $|z-z_i|>d$ for every $i=1, 1, \cdots\cdots m$, since $h(z)$ is analytic in the compact set

$$F=\{z:|z|\leq r\} - \bigcup_{i=1}^{m} \{z:|z-z_i|<d\}, \quad \frac{|h'(z)|}{1+|h(z)|^2}$$

a real-valued continuous function on F attains its maximum M on F.

Let $K_1=\max\left\{M+1,\ \left|\dfrac{n_i}{a_{-n_1}}\right|+1,\ \left|\dfrac{n_2}{a_{-n_2}}\right|+1, \cdots\cdots,\ \left|\dfrac{n_m}{a_{-n_m}}\right|+1\right\}$

Then, for every z, $|z|<r$, we have

$$\frac{|h'(z)|}{1+|h(z)|^2}<K_1\leq\frac{K_1}{1-|z|^2}$$

Let $K_2=\max(K_1, 2K)$. Then

$$\frac{|h'(z)|}{1+|h(z)|^2}<\frac{K}{1-|z|^2}$$

for every z in D and $h(z)$ is normal in D.

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