ON SEMI-SIMPLE RINGS AND THEIR COMPLETE MATRIX RINGS

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1. Introduction.

Let $R$ be a ring and $M$ be a right $R$-module. In this paper we consider the class of all large submodules of $M$ and denote their total intersection by $S(M)$. In section 2, we prove $S(M)$ coincides with the sum of all simple submodules of $M$, the largest semi-simple submodule in $M$. Applying this result to an arbitrary ring $R$ whether or not $R$ contains the identity 1, we prove that the complete matrix ring $R_n$ of all $n \times n$ matrices over $R$ is semi-simple if the ring $R$ is semi-simple as a right $R$-module $R_R$. This proof is given in Section 3. We also investigate semi-simple right ideals of $R$ and $R_n$ and study their relations.

2. Preliminaries.

We call a submodule $P$ of $M$ large in $M$ and write $P \leq M$ in case each non-zero submodule of $M$ meets $P$. The aim of this section is to prove that $S(M)$ coincides with the sum of all simple submodules of $M$ and to seek a necessary and sufficient condition for a module to be semi-simple.

First, we introduce the definition:

**DEFINITION.** A submodule $N$ of $M$ is closed if and only if $N$ has no proper large extensions in $M$.

If $M_P \supseteq P_{R_R}$, then $C$ is called a complement submodule of $P$ in $M$ in case $C$ is a submodule which is maximal in the set of all submodules $Q$ such that $Q \cap P = 0$. By Zorn's lemma, if $P \cap A = 0$, then there exists a complement submodule of $P$ in $M$ containing $A$. By a complement submodule we mean a submodule which is a complement submodule of some submodule of $M$. It is easy to see that the closed submodules of a module $M$ coincide with the complement submodules of $M$. By this fact, $P$ is large in $M$ if and only if $P$ meets every non-zero closed submodule of $M$. For, if $P \cap K = 0$, then we can choose a complement (=closed) submodule $C$ of $P$ containing $K$. If $P$ meets every non-zero closed submodule, then $C = 0$, since $C \cap P = 0$ and so $K = 0$. This shows that $P$ is large in $M$. From this we prove the following lemma:

**LEMMA 1.** Let $A$ and $B$ be submodules of $M$. Then $B$ is large in $A$ if and only if there exists a large submodule $P$ of $M$ such that $B = A \cap P$.

**Proof.** Assume that $B \leq A$ and $K$ be a complement submodule of $B$ in $M$. Put $P = B + K$. Since $B \cap (A \cap K) = B \cap K = 0$ and $A \cap K = 0$, $A \cap P = A \cap (B + K) = B \cap (A \cap K) = B$. Let $D$ be a submodule of $M$ with $P \cap D = 0$. Then also $B \cap (K - D) = B \cap (P \cap (K - D)) = B \cap (K - D) = B \cap K = 0$. By maximality of $K, D \leq K$, hence $D = (B + K) \cap D = 0$. Thus
Let $P$ be large in $M$. If $P$ is large in $M$, then $P \cap A$ is large in $A$ for every submodule $A$ of $M$. This proves that $B$ is large in $A$ if $B = A \cap P$ where $P$ is large in $M$.

Let $N$ be any submodule of $M$. We consider $S(N)$ in $N$, that is, the intersection of all large submodules of $N$. Then the following relation holds between $S(N)$ and $S(M)$.

**Theorem 1.** $S(N) = S(M) \cap N$.

**Proof.** By Lemma 1, \( \{ P \cap N : P \subseteq M \} = \{ Q : Q \subseteq N \} \) for any submodule $N$ of $M$. It follows that

\[
S(M) \cap N = \bigcap \{ P : P \subseteq M \} \cap N \\
= \bigcap \{ P \cap N : P \subseteq M \} \\
= \bigcap \{ Q : Q \subseteq N \} \\
= S(N).
\]

Let $f : M \rightarrow M'$ be an epimorphism and $P'$ be large in $M'$. Then $f^{-1}P' \cap A = 0$ implies $P' \cap fA = 0$ so that $0 = fA \subseteq P'$. Thus $A \subseteq f^{-1}fA \subseteq f^{-1}P' \cap A = 0$, so $f^{-1}P'$ is large in $M$. Hence we obtain the following corollary:

**Corollary 1.** (1) Let $f$ be an $R$-homomorphism of $M$ into $M'$. Then $fS(M) \subseteq S(M')$.

(2) If $N$ is a submodule of $M$, then $(S(M) + N) / N \subseteq S(M/N)$.

**Proof.** (1): Let $y = fx$, $x \in S(M)$, and let $Q$ be an arbitrary large submodule of $fM$. Since $f^{-1}Q$ is large in $M$, $x \in f^{-1}Q$ so that $y = fx \in Q$. Hence $fS(M) \subseteq S(fM) \subseteq S(M')$.

(2) is an immediate consequence of (1).

We call a module $M$ is semi-simple if $M$ is a direct sum of simple submodules. It is the same thing to require that each submodule of $M$ is a direct summand of $M$ [1, p.55].

**Corollary 2.** $M$ is semi-simple if and only if $S(M) = M$. Therefore $S(M)$ is the largest semi-simple submodule of $M$.

**Proof.** Assume that $M$ is semi-simple and let $A$ be any non-zero simple submodule of $M$. Then for each large submodule $P$ of $M$, $A \cap P \neq 0$ so that $A = A \cap P \subseteq P$. Thus $A \subseteq S(M)$ and $M = S(M)$. Conversely, if $A$ is a submodule of $M$ and $B$ is any complement submodule of $A$ in $M$, then $A \oplus B$ is large in $M$ and $S(M) = M$ implies $S(M) \subseteq A \oplus B = M$ so that $A$ is a direct summand of $M$. Hence $M$ is semi-simple. By Theorem 1, $S(S(M)) = S(M) \cap S(M) = S(M)$ and $S(M)$ is semi-simple by the above result. If a submodule $P$ is semi-simple, then $P = S(P) = P \cap S(M) \subseteq S(M)$. Therefore $S(M)$ is a semi-simple submodule of $M$ which contains every semi-simple submodule.

Immediately, we have:

**Corollary 3.** The total intersection $S(M)$ of all large submodules of $M$ is the sum of all simple submodules of $M$.

It is easy to give an example for $S(M/S(M)) \neq 0$. But under some conditions we can get $S(M/S(M)) = 0$. If $M = S(M)$, it is clear. Now assume that $M \neq S(M)$ and we prove $S(M/S(M)) = 0$ if $S(M)$ is closed in $M$. Let $P$ be a simple submodule of $M = M/S(M)$. Since there is a 1:1 correspondence between submodules of $M$ and submodules of $M$ containing $S(M)$, either $P = S(M)$ or there are no submodules between $P$ and $S(M)$.
where $P$ is an inverse image of $\bar{P}$ by a projection map. If $S(M)$ is not large in $P$, then, since $S(P) = P \cap S(M) = S(M)$, $P$ is the only submodule which is large in $P$, contradicting to $S(P) = S(M)$. So $S(M)$ is large in $P$. Thus we have the following:

**Corollary 4.** If $M$ is a module in which $S(M)$ is closed, then $S(M/S(M)) = 0$.

### 3. Semi-simple rings.

We now turn our attention to a ring $R$ regarded as right $R$-module $R^n$. We call a right ideal $K$ (hence a right $R$-module) of $R$ simple in case the only right ideals of $R$ contained in $K$ are $0$ and $K$ itself; $K$ is semi-simple if it is the sum of simple right ideals. In this section we characterize simple right ideals and semi-simple right ideals of a ring $R$ with the identity $1$ and of the complete matrix ring $R_n$ of all $n \times n$ matrices over $R$. Using these results and applying the results obtained in Section 2, we prove that for any ring $R$ (whether or not $R$ contains $1$) $S(R_n) = (S(R))_n$ and also prove that if a ring $R$ is semi-simple as a right $R$-module $R^n$, then so is its complete matrix ring $R_n$. First we consider a ring $R$ with the identity $1$. To avoid the complexity we employ the following notations: For each right ideal $K$ of $R$, and each $p = 1, 2, \ldots, n$, write

$$K(p) = \{A = (a_{ij}) \in R_n : a_{ij} = 0 \text{ if } i \neq p, a_{pj} \in K, j = 1, 2, \ldots, n.\}$$

and for each right ideal $K$ of $R_n$, and each $p$, put $K_{(p)}$ as follows:

$$K_{(p)} = \{a \in R : a = a_{p1} \text{ for some } A = (a_{ij}) \text{ in } K\}.$$

First, we prove that $K_{(p)}$ and $K_{(p)}$ are right ideals of $R_n$ and $R$ respectively.

**Lemma 2.** For each $p = 1, 2, \ldots, n$, $K_{(p)}$ and $K_{(p)}$ are right ideals of $R_n$ and $R$ respectively. Furthermore $K_n = \sum_{p=1}^{n} K_{(p)}$ and $K = \sum_{p=1}^{n} (K_{(p)})_n$.

**Proof.** We denote the matrix units of $R_n$ by $E_{ij}$. Let $A = (a_{ij})$ and $B = (b_{ij})$ in $K_{(p)}$ and $C = (c_{ij})$ be an arbitrary element of $R_n$. Then $A - B = (a_{ij} - b_{ij})$ and $a_{ij} - b_{ij} = 0$ if $i \neq p$ and $a_{pj} - b_{pj} \in K$ for each $j$, so that $K_n(p)$ is closed under subtraction. For each $r, s = 1, 2, \ldots, n$, $A(c_{rs} E_{rt}) = (\sum_{i=1}^{n} a_{ij} E_{ij})(c_{rs} E_{rt}) = \sum_{i=1}^{n} a_{ij} c_{rs} E_{rt}$ is a matrix whose $i$-th rows are all zero if $i \neq p$ and $a_{pj} c_{rs} \in K$. But $AC$ is a sum of such matrices, and therefore $AC \in K_n(p)$. This proves $K_n(p)$ is a right ideal in $R_n$. Furthermore, it is easy to check $K_n = \sum_{p=1}^{n} K_{(p)}$. Next we will show that $K_{(p)}$ is a right ideal in $R$ $(p = 1, 2, \ldots, n)$ and $K = \sum_{p=1}^{n} (K_{(p)})_n$. Since $K$ is closed under addition (and subtraction), the same is true for $K_{(p)}$. Let $a$ in $K_{(p)}$, and $r \in R$. Then by definition of $K_{(p)}$, there exists a matrix $A = \sum a_{ij} E_{ij}$ in $K$ with $a_{p1} = a$. Since a matrix $A(r E_{11}) = \sum a_{ij} r E_{11}$ is in $K$ and its $(p, 1)$-position element is $a_{p1} r = ar, ar \in K_{(p)}$. Thus $K_{(p)}$ is a right ideal in $R$. Let $A = \sum a_{ij} E_{ij}$ be any element of $K$. Then for any $q = 1, 2, \ldots, n$, $B = AE_{q1} = (\sum a_{ij} E_{ij}) E_{q1} = \sum a_{iq} E_{q1}$ is a matrix in $K$ whose $(p, 1)$-position element is $a_{pq}$. This is true for each $p = 1, 2, \ldots, n$, and therefore $a_{pq} \in K_{(p)}$ for each $q$. Since $a_{pq} E_{pq} \in (K_{(p)})_n$ and $A = \sum_{p=1}^{n} a_{pq} E_{pq} \in (K_{(p)})_n$, it
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follows that $K$ is contained in $\sum_{i=1}^{n}(K_{i})_{n}$. This completes the proof of lemma.

**Theorem 2.** If $K$ is a simple right ideal of $R$, then $K(p)$ is a simple right ideal of $R_{n}$ for each $p=1,2,\ldots,n$, and therefore $K_{n}$ is semi-simple in $R_{n}$.

**Proof.** Let $N$ be a right ideal of $R_{n}$ such that $N \subseteq K_{n}(p)$. Then $N(p)$ is a right ideal of $R$ satisfying $(N(p))_{n}(p)=N$. For, if $A=(a_{ij}) \subseteq N$, then $AE_{ij}=\sum_{i}a_{ij}E_{i1}=a_{p1}E_{p1} \subseteq N$ and $a_{p1} \subseteq N(p)$ for each $j$. It follows that $A \subseteq (N(p))_{n}(p)$ and hence $N \subseteq (N(p))_{n}(p)$. Suppose now that $A \subseteq (N(p))_{n}(p)$ and let us show that $A \subseteq N$. Let $a=a_{p1}$ be an element in the $(p,j)$-position of $A$. Then there exists a matrix $B=(b_{ij}) \subseteq N$ with $b_{p1}=a$. Since $BE_{ij}=\sum_{i}b_{ij}E_{i1}=aE_{p1} \subseteq N$, $A=\sum_{i}a_{p1}E_{p1} \subseteq N$ and so $(N(p))_{n}(p)=N$. Since $N \subseteq K_{n}(p)$, $N(p) \subseteq K$ and since $K$ is simple, either $N(p)=0$ or $N(p)=K$. i.e., $N=0$ or $N=K_{n}(p)$. This proves that $K_{n}(p)$ is simple and since $K_{n}=\sum_{i=1}^{n}(K_{i})_{n}(p)$ is a direct sum of simple right ideals, $K_{n}$ is semi-simple.

Now the following lemma can be proved straightforwardly, so the proof will be omitted.

**Lemma 3.** If $K=\sum_{i=1}^{\infty}K_{i}$ is a sum of right ideals of $R$, then $K_{n}(p)=(\sum_{i=1}^{\infty}(K_{i})_{n}(p)=\sum_{i=1}^{\infty}(K_{i})_{n}(p)$.

**Corollary 5.** If $K$ is semi-simple in $R$, then so is $K_{n}(p)$ for each $p=1,2,\ldots,n$.

**Proof.** Write $K=\sum_{i=1}^{\infty}K_{i}$ where $K_{i}$ is simple in $R$. Then by Theorem 2, for each $i \in I$, $(K_{i})_{n}(p)$ is a simple right ideal of $R_{n}$. Since $K_{n}(p)=\sum_{i=1}^{\infty}(K_{i})_{n}(p)$ is a sum of simple right ideals, $K_{n}(p)$ is semi-simple for each $p=1,2,\ldots,n$.

Since, for each right ideal $K$ of $R$, we have $K_{n}=\sum_{p=1}^{n}K_{n}(p)$, we obtain the following corollary:

**Corollary 6.** If $K$ is semi-simple in $R$, then so is $K_{n}$ in $R_{n}$.

**Theorem 3.** If $R$ is a ring with the identity 1, then $(S(R))_{n}$ is semi-simple in $R_{n}$.

**Proof.** Write $S(R)=\sum_{i=1}^{\infty}K_{i}$ where $K_{i}$ are simple right ideals of $R$. Then $(S(R))_{n}(p)=$ $\sum_{i=1}^{\infty}(K_{i})_{n}(p)$ and each $(K_{i})_{n}(p)$ is simple by Theorem 2, so that $(S(R))_{n}(p)$ is semi-simple. But $(S(R))_{n}=$ $\sum_{i=1}^{\infty}(S(R))_{n}(p)$ is a sum of semi-simple right ideals in $R_{n}$, and therefore $(S(R))_{n}$ is semi-simple.

We know that $S(M)$ is the largest semi-simple submodule of $M$ by Corollary 2. Therefore $(S(R))_{n}$ is contained in $S(R_{n})$ by the above result. To prove the converse inclusion, we need the following lemma:

**Lemma 4.** If $K$ is a simple (resp. large) right ideal of $R_{n}$, then there exists a semi-simple (resp. large) right ideal $K$ of $R$ such that $K_{n} \subseteq K_{w}$.

**Proof.** Consider a right ideal $K_{(p)} \subseteq \{a \in R : a=a_{p1} \text{ for some } A=(a_{ij}) \subseteq K\}$ and let $K=$
Then by Lemma 2, \( K \) is a right ideal of \( R \) such that \( K \subseteq K_n \). First assume that \( K \) is simple and we show that \( K_{(p)} \) is simple in \( R \). For this purpose, let \( N_{(p)} \) be a right ideal of \( R \) such that \( N_{(p)} \subseteq K_{(p)} \) and let \( N=(N_{(p)})_n+\sum_{i \neq p} R_{(i)} \), that is, any matrix \( A=(a_{ij}) \) in \( N \) is of the form: for each \( j=1, 2, \ldots, n \), \( a_{pj} \in N_{(p)} \) and if \( i \neq p \), then \( a_{ij} \) is an arbitrary element of \( R \). We note that \( N \cap K=\{A \in K : A=(a_{ij}), a_{pj} \in N_{(p)} \text{ for each } j\} \). Since \( K \) is simple, it follows that \( N \cap K=K \) or 0 and so \( N_{(p)}=K_{(p)} \) or \( N_{(p)}=0 \), that is, \( K_{(p)} \) is simple for each \( p=1, 2, \ldots, n \). Thus \( K \) is a semi-simple right ideal of \( R \) such that \( K \subseteq K_n \). If \( K \) is large in \( R_n \), then \( K \) is also large in \( R \) since \( K \subseteq K_n \). For, if \( P \) is a right ideal of \( R \) such that \( K \cap P=0 \), then \( (K \cap P)_n=K \cap P_n=0 \) so that \( P_n=0 \) and \( P=0 \). This completes the proof of lemma.

The following result is an immediate consequence of Lemma 4 and Theorem 3.

**Corollary 7.** If \( R \) is a ring with the identity 1, then \( S(R_n)=(S(R))_n \).

Now we prove the following theorem which is a generalization of the above result.

**Theorem 4.** For any ring \( R \), \( S(R_n)=(S(R))_n \).

**Proof.** If \( 1 \in R \), then it is through. If \( 1 \notin R_n \), then we imbed \( R \) into the ring \( R' \) with the identity 1 as an ideal and by the case already proved we have \( S(R')=S(R')_n \). Theorem 1 then shows that \( S(R)=S(R') \cap R \). Since \( R_n \) is an ideal in \( R_n \), we can again apply Theorem 1 and obtain

\[
S(R_n)=R_n \cap (S(R'))_n=(R \cap S(R'))_n=(S(R))_n.
\]

This completes the proof of the theorem.

By the above theorem, we can prove the following theorem which is the main result of this section.

**Theorem 5.** If a ring \( R \) is semi-simple as a right \( R \)-module \( R_n \), then so is \( R_n \).

**Proof.** Theorem 4 ensures that \( S(R_n)=(S(R))_n=R_n \) if \( R \) is semi-simple. Therefore \( R_n \) is also semi-simple by Corollary 2.

**References**


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