ON THE CATEGORY OF VECTOR LATTICES

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1. Introduction

Let $E$ be a vector lattice and $M$ be a lattice ideal of $E$. Then we obtain the quotient vector lattice $E/M$ and a lattice homomorphism $p:E\to E/M$. The purpose of the present paper is to show that

(a) Every lattice homomorphism on a vector lattice $E$ with kernel $M$ has an image isomorphic to $E/M$, and

(b) $p:E\to E/M$ is a universal element for a suitable functor (cf. Theorem 2).

For the terminologies used in the present paper we refer to the papers [3] and [4].

2. Preliminaries

An ordered vector space is a real vector space $E$ equipped with a transitive, reflexive, antisymmetric relation satisfying the following conditions:

1. If $x, y, z$ are elements of $E$ and $x \leq y$, then $x + z \leq y + z$

2. If $x$ and $y$ are elements of $E$ and $a$ is a positive real number, then $x < y$ implies $ax \leq ay$.

The positive cone (or simply the cone) $K$ in an ordered vector space $E$ is defined by $K = \{x \in E, x \geq 0\}$, where $0$ denote the zero element of $E$. The cone $K$ has the following "geometric" properties:

1. $K + K \subseteq K$,

2. $\alpha K \subseteq K$ for each real number $\alpha > 0$, and

3. $K \cap (-K) = \{0\}$.

If $K$ is a subset of a real vector space $E$ satisfying (c1), (c2) and (c3), then $x \leq y$ if $y - x \in K$ defines an order relation on $E$ with respect to which $E$ is an ordered vector space with positive cone $K$.

A subset $K$ of $E$ containing zero element and satisfying (c1) and (c2) is called a wedge.

**Definition 1.** An ordered vector space $(E, \leq)$ is called a vector lattice if and only if for each $x$ and $y$ in $E$ there is a unique supremum of $x$ and $y$ in $E$.

**Definition 2.** A linear subspace $M$ of a vector lattice $E$ is called a lattice ideal if $y \in M$ whenever $x \in M$ and $|y| \leq |x|$.

**Definition 3.** A linear mapping $f$ on a vector lattice $E$ to another vector lattice $F$ is a lattice homomorphism if and only if $f(x \land y) = f(x) \land f(y)$ for all $x$ and $y$ in $E$.

A one-one (into) lattice homomorphism is called a lattice isomorphism.

If $M$ is a linear subspace of a vector space $E$ ordered by a cone $K$, the image $C = p(K)$ of $K$ under the canonical quotient mapping $p:E\to E/M=F$ is a wedge in $E$. However $C$ is not a cone in general, even if $E$ is a vector lattice and $M$ is a sublattice of $E$. The next result shows that a much better order theoretic correspondence between $E$
and \( F \) is valid if \( M \) is a lattice ideal.

**Proposition 1.** If \( E \) is a vector lattice and \( M \) is a lattice ideal in \( E \), the quotient space \( F = E/M \) is a vector lattice for the order structure determined by the canonical image \( C \) in \( F \) of the cone \( K \).

**Proof.** Refer to [4], p. 37.

By the proposition 1 we obtain a vector lattice \( E/M \). This vector lattice \( E/M \) is called the quotient vector lattice of \( E \) by its lattice ideal. The canonical mapping \( p: E \rightarrow E/M \) may be described as the function assigning to each \( x \in E \) the unique \( M+x \).

In the following proposition 2 the canonical mapping \( p: E \rightarrow E/M \) of vector lattice \( E \) onto the quotient vector lattice \( E/M \) is a lattice homomorphism.

**Proposition 2.** For a vector lattice \( E/M \) the canonical mapping \( p: E \rightarrow E/M \) is a lattice homomorphism, where \( M \) is a lattice ideal in \( E \).

**Proof.** Refer to [1], p. 525.

3. Theorems

The vector lattice \( E/M \) can be characterized by the following theorem of the lattice homomorphism \( p: E \rightarrow E/M \):

**Theorem 1.** For each lattice homomorphism \( \phi: E \rightarrow F \) of vector lattices with \( \phi(M) = 0 \), where \( M \) is a lattice ideal in a vector lattice \( E \), there is a unique lattice homomorphism \( \phi': E/M \rightarrow F \) such that \( \phi = \phi' \circ p \).

**Proof.** The situation in the theorem is as indicated in the diagram below:

\[
\begin{array}{c}
E \\
\phi \downarrow \\
E/M \\
\phi' \downarrow \\
F
\end{array}
\]

Given the lattice homomorphisms \( \phi \) and \( p \), it suffices to find a lattice homomorphism \( \phi' \) in such a way that the diagram commutes.

If we define \( \phi'(M+a) = \phi(a) \), then \( \phi' \) is well defined. In fact, if \( M+a = M+b \) \((a, b \in E)\), then \( M+ (a-b) = 0 \), which implies that \( a-b \in M \). Hence \( \phi(a-b) = 0 \). Since \( \phi \) is linear, \( \phi(a) = \phi(b) \). Therefore, \( \phi'(M+a) = \phi'(M+b) \).

\( \phi' \) carries all the elements of \( M+a \) to a single element \( \phi(a) \) of \( F \). This means that there is a unique function \( \phi' \) on \( E/M \) to \( F \) with \( \phi' \circ p = \phi \).

This function \( \phi' \) is a linear mapping, since for any \( M+a, M+b \) in \( E/M \) \((a, b \in E)\) and for any scalar \( \lambda \),

\[
\phi'[(M+a) + (M+b)] = \phi'[M + (a+b)] = \phi(a + b) = \phi(a) + \phi(b) = \phi'(M+a) + \phi'(M+b),
\]

\[
\phi'[\lambda(M+a)] = \phi(M + \lambda a) = \phi(\lambda a) = \lambda \phi(a) = \lambda \phi'(M+a).
\]

To show that \( \phi' \) is a lattice homomorphism, we need only show that for any \( M+b \)
and \( M + a \) in \( E/M \) \((a, b \in E)\), we have
\[
\phi'[(M+a) \vee (M+b)] = \phi'(M+a) \vee \phi'(M+b).
\]
Since \( \rho \) and \( \phi \) are lattice homomorphisms,
\[
(M+a) \vee (M+b) = \rho(a) \vee \rho(b) = \rho(a \lor b) = M + (a \lor b).
\]
Hence \( \phi'[(M+a) \vee (M+b)] = \phi'[(M+a) \lor (M+b)] = \phi(a \lor b) = \phi'(M+a) \lor \phi'(M+b) = \phi'(M+M) \).
This completes the proof.

For each lattice homomorphism \( \phi: E \to F \) the kernel of \( \phi \) is a lattice ideal of \( E \) (cf., [1], p. 525). Therefore we have the following corollary:

**Corollary.** For the above lattice homomorphism \( \phi': E/M \to F \), \( \phi' \) is a lattice isomorphism if \( \phi: E \to F \) is a lattice homomorphism of vector lattice with kernel \( M \).

**Proof.** To show that \( \phi' \) is a lattice isomorphism we need only show that \( \phi' \) is one-one. If \( \phi'(M+a) = \phi'(M+b) \) for any \( M+a, M+b \in E/M \), then \( \phi(a) = \phi(b) \), which means that \( a - b \in M \). Therefore \( M+a = M+b \). Hence \( \phi' \) is a lattice isomorphism.

We now construct the category of vector lattices. Let the object be a vector lattice and let "hom" be the function with hom \((E, F) = \{ \phi | \phi: E \to F \) is a lattice hom.;

Since each identity \( 1_E: E \to E \) is a lattice homomorphism of vector lattice \( E \), and since the composite of two lattice homomorphisms of vector lattices is again a lattice homomorphism, these data do determine a category of vector lattices.

Let \( X \) be the category of vector lattices and let \( Y \) be the category of sets. For each vector lattice \( F \) and a fixed vector lattice \( E \), we define \( \mathcal{F} \) as follows:
\[
\mathcal{F}(F) = \text{hom}(E, F) = \{ \phi | \phi: E \to F \text{ is a lattice hom.} \};
\]
and for each lattice homomorphism \( f: S \to T(S, T \subseteq X) \),
\[
\mathcal{F}(f): \mathcal{F}(S) \to \mathcal{F}(T)
\]
\[
\mathcal{F}(f) \alpha = f \circ \alpha \quad \text{for any} \quad \alpha \in \mathcal{F}(S).
\]
Then we have a covariant functor \( \mathcal{F} \) from \( X \) into \( Y \).

For a lattice ideal \( M \) of a fixed vector lattice \( E \), we define \( \mathcal{F}_M \) as follows:
\[
\mathcal{F}_M(F) = \{ \phi | \phi: E \to F \text{ is a lattice hom., and } \phi(M) = 0 \};
\]
and for each lattice homomorphism \( f: S \to T(S, T \subseteq X) \),
\[
\mathcal{F}_M(f) = \mathcal{F}(f).
\]
Then \( \mathcal{F}_M \) is a subfunctor of \( \mathcal{F} \).

Therefore we obtain the following result from the above discussions and Theorem 1.

**Theorem 2.** The subfunctor \( \mathcal{F}_M \) of the covariant functor \( \mathcal{F} \): \( X \to Y \) has a universal element \( (\rho, E/M) \).

**References**


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