ON SOME PROPERTIES OF $\varphi$-MIXING PROCESS

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1. Introduction

Mixing theory was introduced probability theory by A. Renyi, in 1958. Subsequently, Billingsley applied it in measure preserving transformation and F. Papangelou extended to Markov processes whose shift transformation is quasi-mixing (Lecture notes in mathematics 31. [5] pp. 272-279.)

From the view of the above mentioned works the theory is applicable in the study of stochastic processes to investigate the independency of random variable and the possibility to obtain the central limit theorem.

In this paper, I obtained some meaningful properties in the field of $\varphi$-mixing process by using H"older and Minkowski’s inequality, Markov chain and stationarity of process.

2. Background and definition

Let $\{X_n\}$, $n=0, \pm 1, \pm 2, \ldots$, be a strictly stationary sequence of random variables, defined on a probability space $(\Omega, \mathcal{B}, P)$, where $\Omega$ is a sample space, $\mathcal{B}$ is a $\sigma$-field of subsets of $\Omega$ and $P$ is a probability measure on $\mathcal{B}$. For $a \leq b$ define $M_{a}^{b}$ as the $\sigma$-field generated by the random variables $\{X_n\}$. Similarly we can define $M_{a}$ and $M_{b}^{\infty}$ by the same way as above.

**Definition.** If for each $k(-\infty < k < \infty)$, $n \geq 1$,

$$E_1 \subseteq M_{-a}^{k}, \ E_2 \subseteq M_{k}^{+a},$$

$$|P(E_1 \cap E_2) - P(E_1) \cdot P(E_2)| \leq \varphi(n) P(E_1)$$ (2-1)

then $\{X_n\} \subseteq M_{-\infty}^{\infty}$ is $\varphi$-mixing process, where $\varphi$ is a nonnegative function of positive integers.

Note. If $P(E_1) > 0$ then (2-1) is equivalent to

$$|P(E_2|E_1) - P(E_2)| \leq \varphi(n)$$ (2-2)

and therefore (2-2) implies that if $\varphi(n)$ is small, $E_1$ and $E_2$ are virtually independent.

In this paper I will use the following lemmas without proofs.
LEMMA I. In Markov process, if the transition matrix \((P_{uv})\) is irreducible and aperiodic, then \(P_{uv}(\cdot)\rightarrow P_v\) (see [6] pp. 51-54).

LEMMA II. If \(f\) and \(g\) are measurable function and if \(|f|^{p}\) and \(|g|^{q}\) are integrable, where \(p>1\) and \(1/p + 1/q = 1\), then \(fg\) is integrable, and for each measurable set \(E\),
\[
\int_E |fg| \leq \left( \int_E |f|^{p} \right)^{1/p} \left( \int_E |g|^{q} \right)^{1/q}
\]

3. If the transition matrix \((P_{uv})\) is irreducible and aperiodic in Markov process, we can prove the following Theorem I.

THEOREM I. Let \(\{X_n\}\) be a stationary Markov process with finite state space. If the transition matrix \((P_{uv})\) is irreducible and aperiodic, for
\[
E_1 = \{X_{k+\cdot}, \ldots, X_k\} \subseteq M_{\cdot, \cdot}
\]
\[
E_2 = \{X_{k+n}, \ldots, X_{k+n}\} \subseteq M_{\cdot, \cdot},
\]
then we have
\[
|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leq \varphi(n)P(E_1) \rightarrow 0 \quad (3-1)
\]

Proof: Let \(P_u\) be a stationary probability in Markov process, and let \(P_{uv}\) be the transition probability. Then we have generally
\[
P\{X_i = u_0, \ldots, X_{i+\cdot} = U_i\} = P_{u_0}P_{u_0U_1}\cdots P_{u_{i-\cdot}U_i}P_{u_i} \quad (3-2)
\]
for each finite sequence \(u_0, u_1, \ldots, u_i\) of state.

Assume that the \(P_u\) are all positive, then we can easily know that
\[
\varphi(n) = \max |P_{uv}/P_v - 1| \quad (3-3)
\]
is finite.

Using (3-2) we have
\[
|P(E_1 \cap E_2) - P(E_1)P(E_2)| \\
\leq |P_{u_0}P_{u_0u_1}\cdots P_{u_{i-\cdot}U_i}P_{U_iV_0}(u)P_{V_0V_1}\cdots P_{V_{i-\cdot}V_i} - P_{u_0}P_{u_0U_1}\cdots P_{u_{i-\cdot}U_i}P_{V_0V_1}\cdots P_{V_{i-\cdot}V_i}| \\
= P_{u_0}\cdots P_{u_{i-\cdot}U_i}|P_{u_0V_0}(u) - P_{V_0}|P_{V_0V_1}\cdots P_{V_{i-\cdot}V_i} \\
\leq |P_{u_0}\cdots P_{u_{i-\cdot}U_i}|P_{u_0V_0}(u) - P_{V_0}|
\]
Thus, by Lemma I
\[
\varphi(n) = \lim_{n \rightarrow \infty} \max |P_{u_0V_0}(u) - P_{V_0}| = 0
\]
then
\[
|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leq \varphi(n)P(E_1) \rightarrow 0
\]

4. Using the Hölder and Minkowski's inequality in \(\varphi\)-mixing process, we get the following result.

THEOREM II. If \(\{X_n\}\) is \(\varphi\)-mixing process, and \(M_{\cdot, \cdot}, M_{\cdot, \cdot, \cdot}\) is the \(\sigma\)-field
generated by \( \{X_n\} \) for \( X \in M_{-\infty}, Y \in M_{-\infty} \) \( (n \geq 0) \),

\[ |E(XY) - E(X)E(Y)| \leq \phi^r(n) \{E|X|_p\}^{1/p} \{E|Y|_q\}^{1/q} \] (4-1)

where

\[ E|X|^p < \infty, \quad E|Y|^q < \infty, \quad 1/p + 1/q = 1 \]

**Proof:** At first, we suppose two cases \( \phi(n) = 0 \) and \( \phi(n) = 1 \).

(i) \( \phi(n) = 0 \). \( X \) and \( Y \) independent to each other, and thus the inequality (4-1) holds.

(ii) \( \phi(n) = 1 \). By Lemma II, the following holds evidently.

\[ |E(XY) - E(X)E(Y)| = |E[X(Y - E(Y))]| \]

\[ \leq \{E|X|^p\}^{1/p} \{E|Y - E(Y)|^q\}^{1/q} \]

\[ \leq 2\{E|X|^p\}^{1/p} \{E|Y|^q\}^{1/q} \]

(iii) the general case. Suppose

\[ X = \sum u_i I_{A_i}, \quad Y = \sum v_j I_{B_j} \]

where \( \{A_i\} \) and \( \{B_j\} \) are finite decomposition of the sample space \( \Omega \) into elements of \( M_{-\infty} \) and \( M_{+\infty} \).

\[ |E(XY) - E(X)E(Y)| = |\sum u_i v_j [P(A_i)P(B_j|A_i) - P(A_i)P(B_j)]| \]

\[ = |\sum u_i P(A_i) \{\sum v_j (P(B_j|A_i) - P(B_j))\}| \]

\[ = |\sum u_i P(A_i)^{1/p} P(B_j)^{1/q} \sum v_j (P(B_j|A_i) - P(B_j))| \]

\[ \leq \{\sum |u_i| P(A_i)\}^{1/p} \{\sum v_j (P(B_j|A_i) - P(B_j))^{1/q}\} \]

\[ \leq 2\{E|X|^p\}^{1/p} \{E|Y|^q\}^{1/q} \]

But for each \( i \),

\[ |\sum v_j (P(B_j|A_i) - P(B_j))|^{1/q} = |\sum v_j (P(B_j|A_i) - P(B_j))^{1/q} (P(B_j|A_i) - P(B_j))^{1/p}|^{1/q} \]

\[ \leq |\sum v_j^{1/q} (P(B_j|A_i) - P(B_j))| \{\sum v_j (P(B_j|A_i) - P(B_j))\}^{1/q} \]

and since

\[ |\sum P(A_i) |\sum v_j|^{1/q} |P(B_j|A_i) - P(B_j)| \leq 2 \quad E|Y|^q \]

and so

\[ |\sum P(A_i) |\sum v_j (P(B_j|A_i) - P(B_j))|^{1/q} \]

\[ \leq \{\sum P(A_i) \sum v_j (P(B_j|A_i) - P(B_j))\}^{1/q} \{\sum P(B_j|A_i) - P(B_j)\}^{1/q} \]

\[ \leq 2\{E|Y|^q\}^{1/q} \{\sum P(B_j|A_i) - P(B_j)\}^{1/q} \]

If \( C_i^+ [C_i^-] \) is the union of \( B_j \) for which \( P(B_j|A_i) - P(B_j) \) is positive (nonpositive), then \( C_i^+, C_i^- \) lie in \( M_{-\infty} \) and hence

\[ |\sum P(B_j|A_i) - P(B_j)| = P(C_i^+|A_i) - P(C_i^+) + P(C_i^-|A_i) - P(C_i^-) \leq 2 \phi(n), \]

and then
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\[
\{ \sum |P(B_j|A_i) - P(B_j) | \}^{1/p} \leq 2^{1/p} \phi(n)^{1/p}
\]

therefore

\[
|E(XY) - E(X)E(Y)| \leq \{ E|X|^q \}^{1/q} 2^{1/q} \{ E|Y|^q \}^{1/q} 2^{1/p} \phi(n)^{1/p}
\]

\[= 2 \phi(n)^{1/p} \{ E|X|^p \}^{1/p} \{ E|Y|^q \}^{1/q} \]

5. **Theorem III.** Let \( \{X_n\} \) be stationary \( \psi \)-mixing process and \( S_n = \sum_{i=1}^{n} X_i \).

if \( E(X_\omega) = 0, \ E(X_\omega^2) < \infty \), then

\[
n^{-1}E(S_n^2) \to E(X_\omega^2) + 2 \sum_{i=1}^{n} E(X_{i}X_{i+1}) \leq 4 E(X_\omega^2) \sum_{i=p}^{n} \phi(i)^{1/2}
\]

**Proof.** If \( \rho_k = E(X_\omega X_k) \) then by stationarity,

\[
E(S_2) = \sum_{i=1}^{n} E(X_i^2) + 2 \sum_{i=1}^{n} E(X_i X_{i+1}) = n \rho_0 + 2 \sum_{i=1}^{n-1} (n-k) \rho_k
\]

therefore

\[
n^{-1}E(S_n^2) = \rho_0 + 2 \sum_{i=1}^{n-1} (1-k/n) \rho_k
\]

\[
\lim_{n \to \infty} n^{-1}E(S_n^2) = \rho_0 + 2 \sum_{i=1}^{\infty} \rho_k = E(X_\omega^2) + 2 \sum_{i=1}^{\infty} E(X_{i}X_{i+1})
\]

by theorem II.

\[
E(X_\omega^2) \leq 2 \rho^{1/2}(0) E(X_\omega^2)
\]

\[
|E(X_\omega X_k)| \leq 2 \rho^{1/2}(k) E(X_\omega^2)
\]

therefore

\[
\lim_{n \to \infty} n^{-1}E(S_n^2) = E(X_\omega^2) + 2 \sum_{i=1}^{\infty} E(X_{i}X_{i+1})
\]

\[\leq 2 \rho^{1/2}(0) E(X_\omega^2) + 4 \sum_{i=1}^{\infty} \rho^{1/2}(n) E(X_\omega^2)
\]

\[= 4 E(X_\omega^2) \sum_{i=1}^{\infty} \rho^{1/2}(k)
\]

**References**


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