A NOTE ON FIBRE BUNDLES

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Let $\xi=(X,p,B)$ be a principal $G$-bundle (§2), where $G$ is a topological group (§1, §2). For a left $G$-space $F$ the relation $(x,y)s=(xs,s^{-1}y)$ defines a right $G$-space structure on $X \times F$, where $(x,y) \in X \times F$ and $s \in G$. We put $X_F = X \times F \mod G$, and we define $p_F : X_F \to B$ by the commutative diagram

$$
\begin{array}{c}
X \times F \\
\downarrow p_X \\
X \\
\downarrow p \\
B \\
\end{array}
\quad \quad \quad
\begin{array}{c}
X_F \\
\downarrow p_F \\
B \\
\end{array}
$$

where $p_X(x,y)=x$ for all $(x,y) \in X \times F$ (§2).

In this paper, we shall prove that the bundle $\xi(F)=\langle X_F, p_F, B \rangle$ is a fibre bundle (§1) under some conditions (Theorem 1 of §3).

1. Fibre bundles

Let $\xi=(X,p,B,F)$ be a bundle with fibre $F$ satisfying local triviality (§1, §2 and §3). Thus there is an open covering $\{U_j\}_{j \in J}$ of $B$ such that for each $j \in J$

$$
\phi_j : U_j \times F \to p^{-1}(U_j)
$$

is a homeomorphism. $\{U_j\}_{j \in J}$ is called a system of coordinate neighborhoods, and each $\phi_j$ is called the coordinate function. The coordinate functions are required to satisfy the conditions:

$$
p\phi_j(b,y)=b \quad \text{for} \quad (b,y) \in U_j \times F \quad \text{and for} \quad j \in J.
$$

Sometimes, $(U_j, \phi_j)$ is called a chart of $\xi$ over $U_j$.

Let $F$ be an effective $G$-space (§2), where $G$ is a topological group (that is, $G$ is a group of automorphisms of $F$) (§1). We define a map

$$
\phi_{j,b} : F \to p^{-1}(b)
$$

by

$$
\phi_{j,b}(y)=\phi_j(b,y)
$$
(in the sequel, a map means a continuous map), then for each pair \((i, j) \in J \times J\) and for \(b \in U_i \cap U_j\) \(G\) must satisfy the condition that the homeomorphism

\[ \phi_{j, b}^{-1} \circ \phi_{i, b} : F \to F \]

coincides with the operation of a unique element of \(G\). In this case, the group \(G\) is called structure group of the bundle \(\xi\).

Thus, the map \(g_{j, i} : U_i \cap U_j \to G\) defined by

\[ g_{j, i}(b) = \phi_{j, b}^{-1} \circ \phi_{i, b} \]

is continuous. We have the following results:

(i) For any \((i, j, k) \in J \times J \times J\)

\[ g_{k, j}(b) g_{i, j}(b) = g_{k, i}(b), \quad b \in U_i \cap U_j \cap U_k. \]

(ii) For \(i \in J\)

\[ g_{i, i}(b) = \text{the identity of } G, \quad b \in U_i \]

(iii) In (i), put \(i = k\), then from (ii) we obtain

\[ g_{j, k}(b) = (g_{k, j}(b))^{-1}, \quad b \in U_i \cap U_k. \]

If we define the map \(p_j : p^{-1}(U_j) \to F\) by

\[ p_j(x) = \phi_{j, b}^{-1}(x), \]

where \(p(x) = b\), the following identities hold.

(iv) \(p_j \phi_j(b, y) = y, \quad \phi_j(p(x), p_j(x)) = x, \quad g_{j, i}(\phi_j(x)) p_j(x) = p_j(x),\]

where \((b, y) \in U_j \times F, \quad x \in X\) and \(p(x) \in U_i \cap U_j, \quad \{g_{i, j}(i, j) \in J \times J\}\) is called a system of transition functions of \(B\) relative to an open covering \(\{U_j\}_{j \in J}\). In this case, for \(b \in U_i \cap U_j\), we have

\[ \phi_j(b, y) = \phi_i(b, g_{i, j}(b)y). \]

The bundle \(\xi = (X, p, B, F, G)\) is called a coordinate bundle with charts \((U_j, \phi_j)\) and the structure group \(G\).

Two coordinate bundles \(\xi\) and \(\xi'\) are said to be equivalent in the strict sense if they have the same bundle space, base space, projection, fibre and groups, and their charts \(\{U_j, \phi_j\}\), \(\{U'_k, \phi'_k\}\) satisfy the conditions that

\[ \tilde{g}_{k, j}(b) = \phi'_k \tilde{\phi}_j(b), \quad b \in U_j \cap U'_k, \]

coincides with the operation of an element of \(G\), and the map

\[ \tilde{g}_{k, j} : U_j \cap U'_k \to G \]

so obtained is continuous.

**Proposition 1.** The above relation is a proper equivalence relation.

**Proof.** By definition of \(g_{j, i}\), reflexivity is obvious. For

\[ \tilde{g}_{k, j}(b) = \phi'_k \tilde{\phi}_j(b), \quad b \in U_j \cap U'_k, \]

\[ \tilde{g}_{j, i}(b) = \phi_j \tilde{\phi}_i(b), \quad b \in U_i \cap U'_j, \]

\[ \tilde{g}_{i, k}(b) = \phi_i \tilde{\phi}_k(b), \quad b \in U_i \cap U'_k. \]
which is in $G$,

$$
\tilde{g}_{i,k}(b) = \phi_{j,l}^{-1} \cdot \phi_{j,k}^{-1}
$$

is $(\tilde{g}_{i,j}(b))^{-1}$ by the above (iii). Therefore we have the commutative diagram

$$
\begin{array}{ccc}
U_j \cap U'_k = U'_k \cap U_j & \xrightarrow{\tilde{g}_{i,k}} & G \\
\circ & & \Downarrow (\cdot)^{-1} \\
\tilde{g}_{i,j} & & G
\end{array}
$$

where $(\cdot)^{-1} : G \rightarrow G$ is defined by $(\cdot)^{-1}(g) = g^{-1}$, $g \in G$. Since $G$ is a topological group, $(\cdot)^{-1}$ is continuous, and therefore $\tilde{g}_{i,k}$ is continuous. Symmetry is proved.

Assume that $\xi$ is equivalent in the sense to $\xi'$ and $\xi''$ is equivalent in the sense to $\xi'''$. We want to prove that for all $b \in U_j \cap U''_l$

$$
\tilde{g}_{i,j} : U_j \cap U''_l \rightarrow G
$$

is continuous at $b$, where $\tilde{g}_{i,j}(b) = \phi_{j,l}^{-1} \cdot \phi_{j,b}$. Take $U'_k$ such that $b \in U_j \cap U'_k \cap U''_l$, then

$$
\tilde{g}_{k,j} : U_j \cap U'_k \rightarrow G, \quad \tilde{g}_{r,k} : U'_k \cap U''_l \rightarrow G
$$

are continuous. Since

$$
\tilde{g}_{i,k}(b) \tilde{g}_{k,j}(b) = (\phi_{j,l}^{-1} \cdot \phi_{j,k}^{-1}) \cdot (\phi_{l,j}^{-1} \cdot \phi_{j,b}) = \phi_{j,l}^{-1} \cdot \phi_{j,b}
$$

and $G \times G \rightarrow G$ defined by $(g, g') \mapsto gg'$ is continuous, $\tilde{g}_{i,j}$ is continuous at $b$. Therefore transitivity is verified. q.e.d.

With this notion of equivalence, a fibre bundle is defined to be an equivalence class of coordinate bundles. Thus a fibre bundle may regard a maximal coordinate bundle having all possible coordinate functions of equivalence class ([3], [4]).

2. Principal $G$-bundles

Let $X$ be a topological space, and let $G$ be a topological group. $X$ is a right $G$-space if a map $X \times G \rightarrow X$ defined by $(x, s) \mapsto xs \in X$ satisfies the following conditions:

(i) For each $x \in X$, $s \in G$, the relation $x(st) = (xs)t$ holds.

(ii) For each $x \in X$, the relation $x1 = x$ holds, where 1 is the identity of $G$.

A right $G$-space $X$ is said to be effective if it has the property that $xs = x$ implies $s = 1$. Let $X^*$ be the subspace consisting of all $(x, xs) \in X \times X$, where
for an effective $G$-space $X$. There is a function $\tau : X^* \to G$ such that $x \tau (x, x) = xs$ for all $(x, xs) \in X^*$. The function $\tau : X^* \to G$ is called the translation function. From the definition of $\tau$ we have the following:

(iii) $\tau (x, x) = 1$
(iv) $\tau (x, x') \tau (x', x'') = \tau (x, x'')$
(v) $\tau (x', x) = (\tau (x, x'))^{-1}$ for $x, x', x'' \in X$.

A right $G$-space $X$ is said to be principal if $X$ is a right effective $G$-space with a continuous translation function $\tau : X^* \to G$. A principal $G$-bundle is a $G$-bundle $(X, p, B)$, where $X$ is principal.

Let $\xi = (X, p, B)$ be a principal $G$-bundle, and let $F$ be a left $G$-space. The bundle $\xi (F) = (X_F, p_F, B)$ is called the associated bundle of $\xi$ with fibre $F$ (see the first part of this paper). The group $G$ is called the structure group of $\xi (F)$.

**Proposition 2.** In $\xi (F) = (X_F, p_F, B)$, $p_{F^{-1}} (b)$ is homeomorphic to $F$ for all $b \in B$.

**Proof.** Note that there is the translation map $\tau : X^* \to G$ of the $G$-bundle $\xi = (X, p, B)$. Let $p(x_0) = b$ for some $x_0 \in X$. We define the map $f : F \to X_F$ by $f(y) = (x_0, y) G$ for $y \in F$, where $(x_0, y) G$ is an element of $X_F$. Since $p_F((x_0, y) G) = p(x_0) = b$, $f(F)$ is a subset of $p_{F^{-1}} (b)$. Define $g_1 : p^{-1} (b) \times F \to F$ by $g_1 (x, y) = \tau (x_0, x) y$, where $x = x_s$ for some $s \in G$. Then $g_1 (xs, y x') = g_1 (x, y)$. If the map $g : p_{F^{-1}} (b) \to F$ is defined by the commutative diagram

![Diagram]

we know that $f$ and $g$ are inverse to each other. q.e.d.

3. **The Main Theorem**

An atlas of charts of a bundle $\xi = (X, p, B, F)$ with fibre $F$ is a family $\{(U, \phi_j)\}_{j \in J}$ of charts such that $\bigcup_{j \in J} U_j = B$.

**Lemma 1.** For the product bundle $X \times G$, $\xi = (X \times G, p, X)$ there is a one-to-one correspondence between all $X$-automorphisms $\xi \to \xi$ over $X$ and all maps $X \to G$, where $G$ is a topological space. That is, an $X$-automorphism $\phi : \xi$
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$\xi$ corresponds to the map $g : X \rightarrow G$ such that $\phi_t(x, s) = (x, g(x)s)$ for $(x, s) \in X \times G$.

Proof. Define $(X \times G) \times G \rightarrow X \times G$ by $((x, s), t) \mapsto (x, s)t = (x, st)$. Then $X \times G$ is a right $G$-space. Since $\phi_t : \xi \rightarrow \xi$ is an $X$-automorphism we have the commutative diagram:

$$
\begin{array}{ccc}
X \times G & \overset{\phi_t}{\longrightarrow} & X \times G \\
\downarrow p & & \downarrow p \\
X & & X
\end{array}
$$

where $\phi_t$ is a $G$-morphism $([1])$. From $p\phi_t = p$, we have $\phi_t(x, s) = (x, f(x, s))$ for some map $f : X \times G \rightarrow G$. Put $g(b) = f(b, 1)$. Then we have $\phi_t(x, s) = \phi_t(x, 1)s = (x, g(x)s)s = (x, g(x)s)$.

Conversely, from the relation $\phi_t(x, st) = (x, g(x)s)t = \phi_t(x, s)t$, it follows that $\phi_t$ is an automorphism with inverse morphism $\phi_t^{-1} = \phi_t^\prime$, where $g^\prime(x) = g(x)^{-1}$ for $x \in X$.

Lemma 2. Let $\xi = (X \times G, p, X)$ be a right $G$-bundle $([1])$, and let $F$ be a left effective $G$-space. The bundle automorphisms $\xi(F) \rightarrow \xi(F) = (X \times F, q, X)$ are all of the form $\phi_t(x, y) = (x, g(x)y)$, where $g : X \rightarrow G$ is a map and $X \times F = X \times G \times F \mod G$.

Proof. By Lemma 1, our bundle automorphisms are quotients of $(x, s, y) \mapsto (x, g(x)s, y)$. Since $(x, g(x)s, y) = (x, g(x)y) \mod G$, our bundle automorphisms are of the form $(x, y) \mapsto (x, g(x)y) = \phi_t(x, y)$. Since $g(x) \subseteq G$, $g(x)$ is an automorphism of $F$. q.e.d.

Proposition 3. Let $\xi = (X, p, B)$ be a principal $G$-bundle, and let $\xi(F)$ be the associated bundle of $\xi$ with fibre $F$. If $(U, \phi_1)$ and $(U, \phi_2)$ are charts of $\xi(F)$ over $U \subseteq B$, then there is a unique map $g : U \rightarrow G$ such that $\phi_1(b, y) = g(b, g(b)y)$ for each $(b, y) \in U \times F$, where $g(b)$ is an automorphism of $F$.

Proof. By the hypothesis $\phi_2^{-1} \cdot \phi_1 : U \times F \rightarrow U \times F$. By Lemma 2, the automorphism $\phi_2^{-1} \cdot \phi_1$ has the unique map $g : U \rightarrow G$ such that $\phi_2^{-1} \cdot \phi_1(b, y) = (b, g(b)y)$. Since $F$ is a left effective $G$-space $g(b) \subseteq G$ is an automorphism of $F$. In this case, we have $\phi_1(b, y) = \phi_2(b, g(b)y)$. q.e.d.

Suppose the associated bundle $\xi(F) = (X_F, p_F, B)$ of a principal $G$-bundle $\xi =
(X, p, B). Let \{(U_i, \phi_i)\}_{i \in I} be an atlas of \(\xi(F)\). Then \((U_i \cap U_j, \phi_i |_{U_i \cap U_j})\) and \((U_i \cap U_j, \phi_j |_{U_i \cap U_j})\) are charts of \(\xi(F)\) over \(U_i \cap U_j\). For \(b \in U_i \cap U_j\) we define

\[
\phi_i : F \rightarrow \mathbb{p}^{-1}(b) \\
\phi_j : F \rightarrow \mathbb{p}^{-1}(b)
\]

and

\[
\phi_{i,b} : F \rightarrow \mathbb{p}^{-1}(b) \\
\phi_{j,b} : F \rightarrow \mathbb{p}^{-1}(b)
\]

Then, by Proposition 3 there is a unique map

\[g_{j,i} : U_i \cap U_j \rightarrow G\]

such that \(g_{j,i}(b) = \phi_{i,b}^{-1} \phi_{j,b}\) where \(g_{j,i}(b)\) is an automorphism of \(F\). If all \(g_{j,i}(b)\) for \(b \in U_i \cap U_j\) and \((i, j) \in J \times J\) are in \(G\), then \(\{g_{j,i}\}_{(i, j) \in J \times J}\) is a system of transition functions of \(B\) relative to \(\{U_i\}_{i \in I}\), because of we can easily prove that \(g_{j,i}\) satisfies the properties (i)-(iii) in § 1.

THEOREM 1. (Main theorem) Let \(\xi(F)\) be the associated bundle with fibre \(F\) of a principal \(G\)-bundle \(\xi = (X, p, B)\). If \(\xi(F)\) has an atlas \(\{(U_i, \phi_i)\}_{i \in I}\), then \(\xi(F)\) is a fibre bundle.

Proof. We have already proved that \(\xi(F)\) is a coordinate bundle. Therefore we have to show that two coordinate bundles \(\xi(F)\) with atlas \(\{(U_i, \phi_i)\}\) and \(\xi(F)\) with atlas \(\{(U'_i, \phi'_i)\}\) relative to \(G\) are equivalent in the strict sense. Since \((U_i \cap U'_i, \phi_i |_{U_i \cap U'_i})\) and \((U_j \cap U'_j, \phi'_j |_{U_j \cap U'_j})\) are charts of \(\xi(F)\) over \(U_i \cap U'_i\) by Proposition 3\(\xi(F), \{(U_i, \phi_i)\}\) and \(\xi(F), \{(U'_i, \phi'_i)\}\) are equivalent in the strict sense (§ 1). q.e.d.

References


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