A REPRESENTATION OF THE LIE GROUP Aut(G)

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A Lie group G is a manifold (\([1]\)) and a topological group with the map

\[
G \times G \longrightarrow G
\]

\[
(g, h) \longmapsto gh^{-1},
\]

which is a \(C^\infty\)-class function (\([3]\)).

Let \(G\) be a Lie group, and let Aut(\(G\)) be the set of all automorphisms of \(G\). Then Aut(\(G\)) is also a Lie group (\(\S 3\)). Let us denote the Lie algebra of \(G\) by \(\mathfrak{g}\). In this case \(\mathfrak{g}\) is a vector space. Thus, Aut(\(\mathfrak{g}\)) is a matrix group.

Let \(G\) be a connected Lie group. In this paper, we shall prove that the homomorphism

\[
\text{Aut}(G) \longrightarrow \text{Aut}(\mathfrak{g})
\]

\[
dj
\]

(for \(dj\) see \(\S 1\)) is a monomorphism by using exponential maps (\(\S 2\)). Theorem 2 will be the main theorem of this paper.

1. Lie Groups

Let \(M^n\) be an \(n\)-dimensional real manifold (in the sequel, a manifold means a real manifold). For each \(x \in M\) the tangent vector space \(M^n_x\) is an \(n\)-dimensional vector space \([3]\). Given a morphism \(\phi : M \rightarrow N\) (\([1]\)) between two manifolds \(M\) and \(N\), for each \(x \in M\) there is a linear map

\[
\mathfrak{d}\phi : M_x \longrightarrow N
\]

\[
L \longmapsto \mathfrak{d}\phi(L),
\]

where for \(g \in F(N, \phi(X))\), \(\mathfrak{d}\phi(L)(g) = L(g \cdot \phi)\), \(L\) is a tangent vector (\([3]\)) at \(x \in M\) and \(F(N, \phi(x))\) the set of all continuous functions from a suitable open neighborhood of \(x\) to \(R\) (the field of the reals). This \(\mathfrak{d}\phi\) is called the differential of \(\phi\).

PROPOSITION 1. For two morphisms \(\phi : M \rightarrow N\) and \(\psi : N \rightarrow P\) of manifolds,

\[
d(\psi \cdot \phi) = d\psi \cdot d\phi.
\]
Proof. Take $h \in F(P, \psi \cdot \phi(x))$, $x \in M$ and $L \subseteq M_x$. Then,
\[
\begin{align*}
&d(\psi \cdot \phi)(L)(h) = L(h \cdot \psi \cdot \phi), \\
&(d\phi \cdot d\psi)(L)(h) = d\phi(d\psi(L)h) = d\phi(L)(h \cdot \phi) = L(h \cdot \psi \cdot \phi).
\end{align*}
\]
q.e.d.

Let $M$ be a manifold. A vector field $A$ is map
\[
A : M \to TM = \bigcup_{x \in M} M_x,
\]
which assigns to each $x$ of $M$ a tangent vector of $M_x$, where $TM$ is a topological space with topology induced by the projection $p : TM \to M(p^{-1}(x) = M_x)$.

For a vector field $A$ and $f \in F(M, x)$ we define
\[
Af : U \to R
\]
where $U$ is a suitable open neighborhood of $x$. If $Af$ is a $C^\infty$-class function, then $A$ is called a $C^\infty$-vector field. For two $C^\infty$-vector fields $A$ and $B$ we define the new $C^\infty$-vector field $(A, B)$ by
\[
(A, B) = B - A - A \cdot B,
\]
which is called the bracket product of $A$ and $B$. In this case, for three $C^\infty$-vector fields $A, B, C$ and for $a_1, a_2, b_1$ and $b_2$ in $R$, the following hold
\[
\begin{align*}
(a_1A + b_2B, C) &= a_1(A, C) + b_2(B, C) \\
(A, b_1B + b_2C) &= b_1(A, B) + b_2(A, C) \\
(A, A) &= 0, \quad [A, B] = -(B, A) \\
[(A, B), C] + [(B, C), A] + [(C, A), B] &= 0 \quad \text{(Jacobi identity)}.
\end{align*}
\]

Consider a morphism $\phi : M \to N$ of manifolds. If two $C^\infty$-vector fields $A$ of $M$ and $B$ of $N$ have the relation
\[
d\phi(A(x)) = (B\phi(x))
\]
for all $x \in M$, then $A$ and $B$ are said to be $\phi$-related.

Let $G$ be a Lie group. For $g \in G$ the left transformation $L_g$ is defined by
\[
L_g : G \to G
\]
\[
h \mapsto gh.
\]
A $C^\infty$-vector field $A$ of $G$ is called a left invariant vector field if for all $g, h \in G$, $dL_g(A(h)) = A(gh)$. We can prove that $A$ is a left invariant vector
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field iff for $g \in G$ $dL_g(A(e)) = A(g)$, where $e$ is the identity of $G$ ([2], [3]).

Also, if $A$ and $B$ are left invariant vector fields, then so is $[A, B]$ ([2], [3]).

The Lie algebra $\mathfrak{g}$ of a Lie group $G$ is a Lie algebra consisting of all left invariant vector fields of $G$. All right invariant vector fields of $G$ form a Lie algebra which is isomorphic to $\mathfrak{g}$ ([3]).

2. Exponential maps

Consider a Lie group homomorphism $\phi : G \to H$. Let $e$ and $e'$ be the identities of $G$ and $H$, respectively. For each left invariant vector field $A$ of $G$ there exists a unique $\phi$-related left invariant vector field $B$ of $H$ ([2], [3]), i.e., $d\phi(A(e)) = B(e')$. Therefore

$$d\phi : \mathfrak{g} \to \mathfrak{h}$$

$$A \mapsto B$$

is a Lie algebra homomorphism, where $\mathfrak{g}$ is the Lie algebra of $G$ and $\mathfrak{h}$ the Lie algebra of $H$. Put

$\text{Hom}(G, H) = \text{the set of all Lie group homomorphisms between } G \text{ and } H$,

$\text{Hom}(\mathfrak{g}, \mathfrak{h}) = \text{the set of all Lie algebra homomorphisms between } \mathfrak{g} \text{ and } \mathfrak{h}$.

Then, if $G$ is simply connected there is a one-to-one correspondence between $\text{Hom}(G, H)$ and $\text{Hom}(\mathfrak{g}, \mathfrak{h})$ such that

$$\text{Hom}(G, H) \longleftrightarrow \text{Hom}(\mathfrak{g}, \mathfrak{h})$$

$$\phi \longleftrightarrow d\phi$$

([2]).

Consider a closed interval $[a, b]$ in $R$ and a manifold $M$. A $C^\infty$-class function $\gamma : [a, b] \to M$ is called a $C^\infty$-curve of $M$ if $\gamma$ is extended to an open interval. For a $C^\infty$-curve $\gamma$ of $M$, define

$$\gamma_*(t) : F(M, \gamma(t)) \to R$$

by $\gamma_*(t)(f) = \left(\frac{d}{dt} f \circ \gamma\right)(t)$. Then $\gamma_*(t)$ is a tangent vector at $\gamma(t)$ ([2], [3]).

Consider a $C^\infty$-vector field and a $C^\infty$-curve $\gamma$. If $\gamma(0) = x \in M$ and $\gamma_*(t) = A(\gamma(t))$, then $\gamma$ is called the integral curve of $A$ starting at $x$. For each $C^\infty$-vector field $A$ and each $x \in M$, it can be verified that there is a unique integral curve of $A$ starting at $x \in M$ (page 27 of [3]).

$R$ is a Lie group with addition and the Lie algebra $\mathfrak{r}$ of $R$ generated by
Let $\mathfrak{g}$ be the Lie algebra of the group $G$. Take an element $A$ in $\mathfrak{g}$. The map

$$d\gamma_A : \mathbb{R} \rightarrow \mathfrak{g}$$

is a Lie algebra homomorphism. Since $R$ is simply connected there is a unique Lie group homomorphism $\gamma_A : \mathbb{R} \rightarrow G$, which is the integral curve of $A$ starting at $e$ ($e$ : the identity of $G$) ([2], [3]).

Define

$$\exp : \mathfrak{g} \rightarrow G$$

by $\exp(A) = \gamma_A(1)$, where $A \in \mathfrak{g}$ and $\gamma_A$ is the integral curve of $A$ starting at $e$, i.e., $\gamma_A(0) = e$ and $\gamma_A(t) = A(\gamma_A(t))$.

**Proposition 2.** For $t \in \mathbb{R}$, $\gamma_A(t) = \exp(tA)$.

**Proof.** We have to prove that $\gamma_A(t) = \gamma_A(1)$, where $tA \in \mathfrak{g}$ and $\gamma_A$ is the integral curve of $tA$ starting at $e$. Define a Lie group homomorphism $L_t : \mathbb{R} \rightarrow \mathbb{R}$ by $L_t(s) = ts$ for $s \in \mathbb{R}$. The Lie algebra homomorphism $dL_t : \mathbb{R} \rightarrow \mathbb{R}$ corresponds to $L_t$. In this case, for $f \in C^\infty(\mathbb{R}, 0)$ and the generator $\frac{d}{ds}$ of $\mathbb{R}$

$$dL_t\left(\frac{d}{ds}\right)f(0) = \frac{d}{ds}f(ts)(0) = t\frac{d}{ds}f(s)(0),$$

and therefore $dL_t\left(\frac{d}{ds}\right)(0) = t\frac{d}{ds}(0)$.

Define $\tilde{\gamma} : \mathbb{R} \rightarrow G$ by the commutative diagram

$$\begin{array}{ccc}
\mathbb{R} & \longrightarrow & G \\
\uparrow \gamma_A & \searrow \gamma_A \\
\mathbb{R} & \longrightarrow & G \\
\end{array}$$

then, for $s \in \mathbb{R}$, $\tilde{\gamma}(s) = \gamma_A(ts)$ and $\tilde{\gamma}$ is a Lie group homomorphism. Thus, $\tilde{\gamma}$ is the integral curve of $d\left(\frac{d}{ds}\right)$ starting at $e$.

By the definition of $\tilde{\gamma}$, $d\tilde{\gamma} = d\gamma_A \cdot dL_t$. Since $d\gamma_A$ is a linear map we have

$$d\tilde{\gamma}\left(\frac{d}{ds}\right) = d\gamma_A\left(t \frac{d}{ds}\right) = td\gamma_A\left(\frac{d}{ds}\right) = tA,$$

and therefore $\tilde{\gamma} = \gamma_A$. i.e. $\gamma_A(s) = \gamma_A(ts)$. q.e.d.

**Corollary 1.** $\exp(t_1 + t_2)A = (\exp t_1 A)(\exp t_2 A)$

$$\exp(-tA) = (\exp tA)^{-1}.$$
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\textbf{Proof.} Since \( \gamma^A : R \rightarrow G \) is a Lie group homomorphism we have
\[
\gamma_A(t_1 + t_2) = \gamma_A(t_1) \gamma_A(t_2)
\]
\[
\gamma_A(-t) = (\gamma_A(t))
\]
Thus, by Proposition 2 our assertion is easily proved. q.e.d.

\textbf{Proposition 3.} Let \( \phi : G \rightarrow H \) be a Lie group homomorphism. Then the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{d\phi} & \mathcal{H} \\
\exp & \downarrow & \exp \\
G & \xrightarrow{\phi} & H
\end{array}
\]

where \( \mathcal{G} \) is the Lie algebra of \( G \) and \( \mathcal{H} \) the Lie algebra of \( H \).

\textbf{Proof.} Let \( A \) be an element of \( \mathcal{G} \), and let \( \gamma_A \) be the integral curve of \( A \) starting at \( e \) (the identity of \( G \)), then \( d\gamma_A\left(\frac{d}{ds}(0)\right) = A(e) \) and \( \phi \cdot \gamma_A : R \rightarrow H \) is a Lie group homomorphism. Moreover, \( \phi \cdot \gamma_A \) is the integral curve of \( d(\phi \cdot \gamma_A)\left(\frac{d}{ds}\right) \) starting at \( e' \) (the identity of \( H \)). Since
\[
d(\phi \cdot \gamma_A)\left(\frac{d}{ds}(0)\right) = d\phi \cdot d\gamma_A\left(\frac{d}{ds}(0)\right) = d\phi(A(e)),
\]
we have
\[
\phi \cdot \exp(A) = \phi \cdot \gamma_A(1) = \exp(d\phi(A)).
\]
q.e.d.

\textbf{3. The main theorem}

Let \( G \) be a Lie group, and let \( \mathcal{G} \) be the Lie algebra of \( G \). If \( G \) is an \( n \)-dimensional manifold, then \( \mathcal{G} \) is an \( n \)-dimensional vector space over \( R \). Thus \( \text{Aut}(\mathcal{G}) \) (the set of all Lie algebra automorphisms of \( \mathcal{G} \)) is isomorphic to a subgroup of \( GL(n, R) \) (the set of all non-singular \( n \times n \) matrices). Note that \( GL(n, R) \) is a Lie group.

Consider a Lie algebra homomorphism \( \phi : \mathcal{G} \rightarrow \mathcal{G} \). \( \phi \) is called a derivation of \( \mathcal{G} \) if it satisfies the condition
\[
\phi(AB) = \phi(A)B + A\phi(B)
\]
for \( A, B \in \mathcal{G} \). Let \( G \) be simply connected. Then it has been proved that \( \text{Aut}(G) \) (the set of all Lie group automorphisms of \( G \)) is isomorphic to a Lie group whose Lie algebra is the algebra of all derivations of \( \mathcal{G} \) (p.138 of [2]). Let \( G \) be connected. Using the above fact and a simply connected covering group of \( G \), the following is proved (for proof see p.138 of [2]).

\textbf{Theorem 1.} Let \( G \) be a connected Lie group. Then \( \text{Aut}(G) \) is a Lie group.
A representation of a Lie group $G$ is a group homomorphism of $G$ into a matrix group. A faithful representation is a representation which is an isomorphism. Thus if a Lie group has a faithful representation, it is isomorphic to a subgroup of matrix group.

**Theorem 2.** Let $G$ be a connected Lie group, and let $\mathfrak{g}$ be the Lie algebra of $G$. The map $\phi: \text{Aut}(G) \rightarrow \text{Aut}(\mathfrak{g})$ defined by $\phi(j) = dj$ for $j \in \text{Aut}(G)$ is a faithful representation of Lie group $\text{Aut}(G)$.

**Proof.** Since $\phi(j \cdot k) = d(j \cdot k) = dj \cdot dk$ for $j, k \in \text{Aut}(G)$, $\phi$ is a group homomorphism. For $j \in \text{Aut}(G)$, consider the following diagram:

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{dj} & \mathfrak{g} \\
\exp & \circ & \exp \\
G & \xrightarrow{j} & G
\end{array}
$$

If for $A \in \mathfrak{g}$ $dj(A) = 0$, then $j(\exp(A)) = \exp(0) = e$. Note that $\exp(0) = \gamma_0$ (1) = $e$. Since $j$ is a Lie group homomorphism, $\exp(A) = e$ and therefore $A = 0$. Therefore $dj$ is in $\text{Aut}(\mathfrak{g})$. Our group homomorphism $\phi$ is thus well defined.

We want to show that $\phi$ is injective. Assume that $j$ is in $\text{Ker}(\phi)$. Then $dj$ is the identity of the group $\text{Aut}(\mathfrak{g})$, and therefore $dj(A) = A$ for all $A \in \mathfrak{g}$. By the above commutative diagram, for all $A \in \mathfrak{g}$ $j(\exp(A)) = \exp(dj(A)) = \exp(A)$. Since every element of $G$ is denoted by $\exp(A)$ for some $A \in \mathfrak{g}$ ([1], [3]), $j$ is the identity map. Thus, $\phi$ is injective. q.e.d.

**References**

