A NOTE ON AN INTERSECTION THEOREM IN BANACH SPACE

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Introduction.

Let $X$ be a real Banach Space. Assume that $X = A \oplus B$, i.e. $X$ is the direct sum of two subspaces $A \subseteq X$ and $B \subseteq X$.

Let $f : A \to X$ and $g : B \to X$ and $h : A \to X$ be three continuous mappings. We have given conditions on $f$, $g$, and $h$ such that $f(A) \cap g(B) \cap h(A) \neq \emptyset$.

1. In [1] Kannan has given a theorem concerning simultaneous fixed points of two mappings in a metric space. We give here a version of the theorem in a Banach Space.

THEOREM A. Let $E$ be a Banach Space. If $T_1$ and $T_2$ are continuous mappings from $E$ into $E$ such that

$$
\|T_1(x) - T_2(y)\| \leq \beta (\|x - T_1(x)\| + \|y - T_2(y)\|)
$$

where $x, y \in E$ and $0 < \beta < \frac{1}{2}$, then $T_1$ and $T_2$ have a common fixed point.

2. An intersection theorem in Banach Space.

By $Q_A : X \to A$ and $Q_B : X \to B$ we denote the projection mappings of $X$ onto $A$ and $X$ onto $B$ respectively. The mappings $Q_A$ and $Q_B$ are linear and we have

$$
\|Q_A x\| \leq \|Q_A\| \|x\|, \quad \|Q_B x\| \leq \|Q_B\| \|x\|
$$

for all $x$ in $X$, where $\|Q_A\|$ and $\|Q_B\|$ are the norms of $Q_A$ and $Q_B$ respectively. Let $F$, $G$ and $H$ are mappings from $X$ into $X$ and define the mappings $T_1$ and $T_2$ as follows:

$$
T_1 = F \circ Q_A - G \circ Q_B,
$$

$$
T_2 = H \circ Q_A - G \circ Q_B.
$$

THEOREM 1. Let $X = A \oplus B$ and let $f : A \to X$, $g : B \to X$ and $h : A \to X$ be such that $f(a) = a - F(a)$, for all $a$ in $A$. 
Proof of theorem 1.

Since $X = A \oplus B$, any element $x$ in $X$ can be written as $x = a - b$ where $a \in A$ and $b \in B$. Since from theorem A, $T_1$ and $T_2$ have a common fixed point which is $x_0$ say with representation $x_0 = a_0 - b_0$, we have,

\[ T_1(x_0) = x_0 \]
\[ \text{i.e. } F \circ Q_A(x_0) - G \circ Q_B(x_0) = a_0 - b_0 \]
\[ \text{i.e. } F(a_0) - G(b_0) = a_0 - b_0 \]
in other words
\[ a_0 - F(a_0) = b_0 - G(b_0) \]
also we have
\[ T_2(x_0) = x_0 \]
\[ \text{i.e. } H \circ Q_A(x_0) - G \circ Q_B(x_0) = a_0 - b_0 \]
\[ \text{i.e. } H(a_0) - G(b_0) = a_0 - b_0 \]
\[ \text{i.e. } a_0 - H(a_0) = b_0 - G(b_0) \]
and therefore
\[ f(A) \cap g(B) \cap h(A) \neq \emptyset. \]

In case $f$, $g$ and $h$ are linear mappings the conditions on $f$, $g$ and $h$ can be given in terms of the norms of the linear mappings $F$, $G$ and $H$ to make sure that $f(A) \cap g(B) \cap h(A) \neq \emptyset$.

THEOREM 2. Let $X = A \oplus B$ and let $f$, $g$ and $h$ are linear continuous mappings as shown in respective subspaces of theorem 1. Then if

\[ \|F\|\|Q_A\| + \|G\|\|Q_B\| < \frac{\beta}{1+\beta} \]
and

\[ \|H\|\|Q_A\| + \|G\|\|Q_B\| < \frac{\beta}{1+\beta} \]

then

\[ f(A) \cap g(B) \cap h(A) \neq \emptyset. \]

Proof of theorem 2.

Proof lies in observing the fact that the conditions in (1) imply that $F \circ Q_A - G \circ Q_B$ and $H \circ Q_A - G \circ Q_B$ satisfy the hypothesis of theorem A. We start with the inequality
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\[ \| F \circ Q_A(x) - G \circ Q_B(x) - H \circ Q_A(y) + G \circ Q_B(y) \| \leq \beta \{ \| x - F \circ Q_A(x) + G \circ Q_B(x) \| + \| y - H \circ Q_A(y) + G \circ Q_B(y) \| \} \]  

(2)

Now left hand side of (2) is

\[ \leq (\| F \|\| Q_A \| + \| G \|\| Q_B \|) \| x \| + (\| H \|\| Q_A \| + \| G \|\| Q_B \|) \| y \| \]

also

\[ \beta \{ \| x - F \circ Q_A(x) \| - \| G \circ Q_B(x) \| \} + \beta \{ \| y - H \circ Q_A(y) \| - \| G \circ Q_B(y) \| \} \leq \text{R.H.S.} \]

since

\[ \| G \circ Q_B(x) \| \leq \| G \|\| Q_B \|\| x \|, \] for all x in X,

we have

\[ \beta \{ \| x - F \circ Q_A(x) \| - \| G \|\| Q_B \|\| x \| \} + \beta \{ \| y - H \circ Q_A(y) \| - \| G \|\| Q_B \|\| y \| \} \leq \text{R.H.S.} \]

from this we get

\[ \beta \{ \| x \| (1 - \| G \|\| Q_B \|) - \| F \|\| Q_A \|) \} + \beta \{ \| y \| (1 - \| G \|\| Q_B \|) - \| H \|\| Q_A \|) \} \leq \text{R.H.S.} \]

therefore

\[ \beta \{ \| x \| (1 - \| G \|\| Q_B \|) - \| F \|\| Q_A \|) \} + \beta \{ \| y \| (1 - \| G \|\| Q_B \|) - \| H \|\| Q_A \|) \} \leq \text{R.H.S.} \]

therefore if

\[ \| F \|\| Q_A \| + \| G \|\| Q_B \| \leq \beta \{ (1 - \| G \|\| Q_B \|) - \| F \|\| Q_A \|) \}

\]

and

\[ \| H \|\| Q_A \| + \| G \|\| Q_B \| \leq \beta \{ (1 - \| G \|\| Q_B \|) - \| H \|\| Q_A \|) \} \]  

(3)

then (2) is satisfied.

The conditions (3) can be written as

\[ \| F \|\| Q_A \| + \| G \|\| Q_B \| \leq \frac{\beta}{1 + \beta} \]

and

\[ \| H \|\| Q_A \| + \| G \|\| Q_B \| \leq \frac{\beta}{1 + \beta} \]

which completes the proof.

**COROLLARY to theorem 2.**

Let \( R \) denote a set of real numbers. Let \( X = A \oplus B \) and \( f : R \times A \to X \), \( g : R \times B \to X \) and \( h : R \times A \to X \) be continuous linear mappings such that

\[ f(\lambda ; a) = a - \lambda F(a) \text{ for all } a \text{ in } A \]

and \( \lambda \) is in \( R \),

\[ g(\lambda ; b) = b - \lambda G(b) \text{ for all } b \text{ in } B \]

\[ h(\lambda ; a) = a - \lambda H(a) \text{ for all } a \text{ in } A. \]

Then if

\[ |\lambda| (\| F \|\| Q_A \| + \| G \|\| Q_B \|) \leq \frac{\beta}{1 + \beta} \]
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\[ |\lambda| \left( \|H\| \|Q_A\| + \|G\| \|Q_B\| \right) \leq \frac{\beta}{1+\beta} \]

\[ f(\lambda : A) \cap g(\lambda : B) \cap h(\lambda : A) \neq \phi. \]

The proof is similar to the proof of theorem 2.

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REFERENCES