BIQUASI-PROXIMITY SPACES AND COMPACTIFICATION
OF A PAIRWISE PROXIMITY SPACE

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1. Introduction. This paper investigates certain properties of a set $X$ equipped with two quasi-proximities. Such a space is called a biquasi-proximity space and is equivalent to a bitopological space, i.e., every bitopological space admits of two quasi-proximities which yield the topologies of the space. Section 2 contains definitions and results that are needed in order to prove the results in the following sections. In section 3 are given characterizations of some separation axioms in bitopological spaces. These characterizations are purely in terms of the two quasi-proximities of the space. When the two quasi-proximities coincide, they reduce to the corresponding results about quasi-proximity spaces [7]. In section 4 we construct a compactification of a pairwise proximity space. Such spaces are equivalent to pairwise completely regular spaces. This compactification reduces to the Smirnov compactification of a proximity space if the quasi-proximities of pairwise proximity space coincide.

2. Definitions and elementary properties.

(a) Proximities. Let $X$ be a non-empty set. A quasi-proximity on $X$ is a relation $\delta$ on the family $\mathcal{P}(X)$ of all subsets of $X$ satisfying the following axioms:

Q.1. $(A, B) \in \delta$ implies $A \neq \emptyset$, $B \neq \emptyset$

Q.2. $(A \cup B, C) \in \delta$ iff $(A, C) \in \delta$ or $(B, C) \in \delta$, and $(A, B \cup C) \in \delta$ iff $(A, B) \in \delta$ or $(A, C) \in \delta$

Q.3. $A \cap B \neq \emptyset$ implies $(A, B) \in \delta$

Q.4. If $(A, B) \notin \delta$, then there exists an $E \in \mathcal{P}(X)$ such that $(A, E) \notin \delta$ and $(X - E, B) \notin \delta$.

A proximity $\delta$ on $X$ is a quasi-proximity $\delta$ on $X$, which is also symmetric, i.e.,

Q.5. $(A, B) \in \delta$ implies $(B, A) \in \delta$.

A (quasi-) proximity $\delta$ is said to be separated if it satisfies the axiom

Q.6. $(\{x\}, \{y\}) \in \delta$ implies $x = y$. 

To each quasi-proximity $\delta$ on $X$ there is associated another quasi-proximity $\delta^{-1}$ on $X$ defined by

$$(A, B) \in \delta^{-1} \text{ iff } (B, A) \in \delta.$$  

$\delta^{-1}$ is called the conjugate of $\delta$. Clearly $(\delta^{-1})^{-1} = \delta$ and $\delta$ is a proximity iff $\delta = \delta^{-1}$.

A subset $C$ of a quasi-proximity space $(X, \delta)$ is said to be closed if $(\{x\}, C) \in \delta$ implies $x \in C$. The topology defined by the collection of all such closed sets is called the topology generated by $\delta$ and is denoted by $T(\delta)$. This topology need not satisfy any separation axiom, but it is necessarily completely regular if $\delta$ is a proximity. Also $T(\delta)$ is $T_1$ iff $\delta$ is separated. Any topological space $(X, T)$ can be equipped with a quasi-proximity $\delta$ such that $T = T(\delta)$ [9, 12]. Thus quasi-proximity spaces are equivalent to topological spaces. Likewise biquasi-proximity spaces are equivalent to bitopological spaces.

Let $(X, \delta)$ be a proximity space. A collection $\sigma$ of subsets of $X$ is called a cluster if the following conditions are satisfied:

(i) If $A$ and $B$ belong to $\sigma$ then $(A, B) \in \delta$
(ii) If $(A, B) \in \delta$ for every $B \in \sigma$, then $A \in \sigma$
(iii) If $A \cup B \in \sigma$, then $A \in \sigma$ or $B \in \sigma$.

For each $x \in X$, the collection

$$\{A \subseteq X : (A, x) \in \delta\}$$

is a cluster, called a point cluster. The concept of cluster is analogue of ultrafilter in topological spaces and is used in constructing Smirnov compactification of a proximity space.

If $\delta$ is a quasi-proximity on $X$, then $\delta^*$ defined by $"(A, B) \in \delta^* \text{ iff for each finite covers } \{A_1, A_2, \ldots, A_m\}, \{B_1, B_2, \ldots, B_n\} \text{ of } A \text{ and } B, (A_i, B_j) \in \delta \text{ and } (B_j, A_i) \in \delta \text{ for some } i = 1, \ldots, m, j = 1, \ldots, n,"$ is a proximity on $X$. It is called the proximity generated by $\delta$.

Other terms about proximities not defined here can be found in [8].

(b) Bitopological Spaces. If $T_1$ and $T_2$ are two topologies on a non-empty set $X$, then the ordered triple $(X, T_1, T_2)$ is called a bitopological space [2]. It is said to be (a) pairwise $T_0$ [6] if for every pair of distinct points there exists a $T_i$ or a $T_2$-neighbourhood of one point not containing the other; (b) pairwise $T_1$ [6] if for every pair of distinct points $x, y$ there exists a $T_1$ or a $T_2$-neighbourhood of $x$ not containing $y$; (c) pairwise $T_2$ [14] if for every pair of
distinct points \( x, y \) there exist a \( T \) neighbourhood of \( x \) and a \( T \)-neighbourhood of \( y \) \((i \neq j)\) which are disjoint; 
(d) pairwise Urysohn \([10]\) if for any two points \( x \) and \( y \) such that \( x \neq y \) there exists a \( T \)-open set \( U \) and a \( T \)-open set \( V \) such that \( x \in U, y \in V, T-cl U \cap T-cl V = \emptyset \) \((i \neq j)\); 
(e) pairwise \( R_0 \) \([6]\) if for every \( G \) in \( T \), \( x \in G \Rightarrow T-cl \{x\} \subset G \) \((i \neq j)\); 
(f) pairwise \( R_1 \) \([7]\) if for \( x, y \in X \), and \( i \neq j \) \( T-cl \{x\} \neq T-cl \{y\} \) implies \( x \) has a \( T \)-neighbourhood and \( y \) has a \( T \)-neighbourhood which are disjoint; 
(g) pairwise \( RO \) \([6]\) if for every \( G \) in \( T \), \( x \in G \) implies \( T-cl \{x\} = \emptyset \) \((i \neq j)\); 
(h) pairwise \( RC \) if every point of \( X \) has a \( T \)-neighbourhood base consisting of \( T \)-closed sets \((i \neq j)\); 
(i) pairwise \( RC \) \([2]\) if for every \( T \)-closed set \( A \) and a \( T \)-closed set \( B \) with \( A \cap B = \emptyset \), there exists a \( T \)-neighbourhood \( F \) of \( A \) and a \( T \)-neighbourhood \( G \) of \( B \) such that \( F \cap G = \emptyset \).

A cover \( Z \) of a bitopological space \((X, T_1, T_2)\) is said to be \textit{pairwise open} \([1]\) if \( Z \subset T_1 \cup T_2 \) and if furthermore contains a non-empty member of \( T \) \((i=1, 2)\). A bitopological space \((X, T_1, T_2)\) is said to be \textit{pairwise compact} \([3]\) if each \( T \)-closed set \( C \neq X \) is \( T \)-compact \((i \neq j)\). A cover \( Z \) of a bitopological space \((X, T_1, T_2)\) is said to be \textit{\( T_1 \)-\( T_2 \)-open} \([13]\) if \( Z \subset T_1 \cup T_2 \). If every \( T \)-\( T \)-open cover of \( X \) has a finite subcover, the bitopological space \((X, T_1, T_2)\) is said to be \textit{compact} \([13]\). Clearly every compact bitopological space is pairwise compact.

Notations. In a quasi-proximity space \((X, \delta)\), \( T(\delta) \)-closure of \( A \subset X \) is denoted by \( \delta-cl A \). \( A \subset B \) means \( (A, X-B) \in \delta \), \( B \) is called a \( \delta \)-neighbourhood of \( A \). \((\{x\}, \{y\}) \in \delta \) is written as \((x, y) \in \delta \). Closure of \( \{x\} \) in \( T(\delta) \) is written as \( x^\uparrow \).

3. Biquasi-proximity Spaces. In this section we give characterizations of some separation axioms in terms of the quasi-proximities of a biquasi-proximity space, without explicit reference to the induced topologies. Proofs of the results are simple and hence many of them are omitted.

THEOREM 3.1 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise \( T_0 \) iff \((x, y) \in \beta \cap \beta^{-1} \Rightarrow x = y\), where \( \beta = \delta_1 \cap \delta_2 \).

THEOREM 3.2 \((X, \delta_1, \delta_2)\) is pairwise \( T_1 \) iff \((x, y) \in \beta \Rightarrow x = y\), where \( \beta = \delta_1 \cap \delta_2 \).

THEOREM 3.3 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise \( R_0 \) iff \((x, y) \in \beta \cap \beta^{-1} \Rightarrow x = y\), where \( \beta = \delta_1 \cap \delta_2 \).
THEOREM 3.4 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise \(T_2\) iff for each pair \(x, y\) of distinct points of \(X\) there exists a covering \(\{A, B\}\) of \(X\) such that \((x, A) \in \delta_i, (y, B) \in \delta_j\) for \(i \neq j, i, j = 1, 2\).

PROOF. The necessity is obvious. For sufficiency, suppose the condition is satisfied, so that for \(x \neq y\), there exist \(A, B\) such that \((x, A) \in \delta_i, (y, B) \in \delta_j\), \(A \cup B = X\). Clearly \(X - A\) and \(X - B\) are the required disjoint neighbourhoods of \(x\) and \(y\) respectively.

THEOREM 3.5 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise \(T_2\) iff \((x, y) \in \delta \Rightarrow (x, y) \in (x, y) \in \Delta \Leftrightarrow x = y\).

PROOF. \((X, \delta_1, \delta_2)\) is pairwise \(T_2\) iff \(\Delta\) is closed in \(\delta\) iff \((x, y) \in \delta \Rightarrow (x, y) \in \Delta \Leftrightarrow x = y\).

THEOREM 3.6 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise \(R_1\) iff for each pair \(x, y\) of points of \(X\) for which there exists a \(p \in X\) such that 
\[(p, x) \in \delta_i \text{ but } (p, y) \notin \delta_j,\]
or 
\[(p, x) \notin \delta_i \text{ but } (p, y) \in \delta_j,\]
there exists a covering \(\{A, B\}\) of \(X\) such that \((x, A) \in \delta_i\) and \((y, B) \in \delta_j\).

THEOREM 3.7 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise Urysohn iff for \(x \neq y\), there exists a pairwise open cover \(\{A_i, A_j\}\) of \(X\) such that \((x, A_i) \in \delta_i, (y, A_j) \in \delta_j\), where \(A_i \in \delta_i\)-open, \(A_j \in \delta_j\)-open, \(i \neq j\).

COROLLARY 3.1 For a biquasi-proximity space \((X, \delta_1, \delta_2)\), \((X, \delta_1, \delta_2)\) is pairwise \(Urysohn \Rightarrow (X, \delta_1, \delta_2)\) is pairwise \(T_2\). \((X, \delta_1, \delta_2)\) is pairwise \(R_1\) \((X, \delta_1, \delta_2)\) is pairwise \(R_0\).

THEOREM 3.8 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise regular iff for each \(x \in X\) and for each \(\delta_i\)-neighbourhood \(A\) of \(x\), there exists a \(\delta_i\)-neighbourhood \(B\) of \(x\) such that \(\delta_i \text{ cl } B \subseteq_i A, i \neq j\).

PROOF. The sufficiency part is obvious. For necessity part, let \(x\) be in \(X\) and let \(A\) be a \(\delta_i\)-neighbourhood of \(x\). Since \((x, X-A) \in \delta_i\), there exists a \(\delta_i\)-neighbourhood \(D\) of \(x\) such that \((D, X-A) \in \delta_i\). Let \(B\) be a \(\delta_j\)-closed neighbourhood of \(x\) contained in \(D\). Then \(B \subseteq_i A\).

THEOREM 3.9 A biquasi-proximity space \((X, \delta_1, \delta_2)\) is pairwise completely regular.
iff there exists a quasi-proximity $\delta$ on $X$ with $\mathcal{F}(\delta) = \mathcal{F}(\delta)$, $\mathcal{F}(\delta) = \mathcal{F}(\delta^{-1})$, $\neq \emptyset$.

PROOF. See [5].

As is well-known there are pairwise normal spaces which are not pairwise completely regular. Therefore the topologies of a pairwise normal space need not be conjugate. But if they are, we have the following result.

**THEOREM 3.10** $(X, \delta, \delta^{-1})$ is pairwise normal iff $\delta$ is defined as $(A, B) \in \delta$ iff $Q$-cl $A \cap P$-cl $B \neq \emptyset$ where $(P, Q)$ is a pair of conjugate topologies on $X$.

PROOF. Clearly $P = \mathcal{F}(\delta)$, $Q = \mathcal{F}(\delta^{-1})$. Now the result follows from [5].

It is known that a compact Hausdorff space admits of a unique proximity. Here we generalize this result to bitopological space.

**LEMMA 3.1** Let $(X, P, Q)$ be a pairwise completely regular space and let $\delta$ be a quasi-proximity on $X$ such that $P = \mathcal{F}(\delta)$, $Q = \mathcal{F}(\delta^{-1})$. If $A$ is $P$-compact and $B$ is $P$-closed, then $A \cap B = \emptyset \Rightarrow (A, B) \in \delta$.

**PROOF.** For each $a \in A$, $(a, B) \in \delta$, so that there exists a $P$-neighbourhood $N_a$ of $a$ such that $(N_a, B) \in \delta$. $\{N_a : a \in A\}$ is a $P$-neighbourhood cover of $A$, and so admits of a finite subcover $\{N_a : i = 1, \ldots, n\}$. Clearly $(A, B) \in \delta$, since $A \subseteq N = \bigcup_{i=1}^{n} N_a$.

**THEOREM 3.11.** For every pairwise compact space $(X, P, Q)$, whose topologies are conjugate there is a unique quasi-proximity $\delta$ on $X$ such that $P = \mathcal{F}(\delta)$, $Q = \mathcal{F}(\delta^{-1})$.

**PROOF.** Let us define $\delta$ on $X$ by setting $(A, B) \in \delta$ iff $(Q$-cl $A) \cap (P$-cl $B) \neq \emptyset$. Then $\delta$ is a quasi-proximity on $X$ such that $P = \mathcal{F}(\delta)$, $Q = \mathcal{F}(\delta^{-1})$. If $\delta_1$ is any other quasi-proximity with this property, then $(Q$-cl $A) \cap (P$-cl $B) \neq \emptyset \Rightarrow (Q$-cl $A$, $P$-cl $B) \in \delta_1 \Leftrightarrow (A, B) \in \delta_1$. Since $Q$-closed sets in a pairwise compact space are $P$-compact, therefore $Q$-cl $A$ is $P$-compact. From lemma 3.1 $(Q$-cl $A) \cap (P$-cl $B) = \emptyset \Rightarrow (A, B) \in \delta_1$. Therefore $\delta_1 = \delta$.

4. **Compactification of a pairwise proximity space.** In this section we shall consider only pairwise $T_1$ biquasi-proximity spaces.

As we have seen in Theorem 3.9 a biquasi-proximity space $(X, \delta, \delta^{-1})$ is pairwise completely regular and all pairwise completely regular spaces are of this form, we shall call such spaces pairwise proximity spaces.
DEFINITION 4.1 Let \((X, \delta_1)\) and \((Y, \delta_2)\) be two quasi-proximity spaces. A function \(f : X \rightarrow Y\) is said to be \textit{proximity mapping} if \((A, B) \in \delta_1 \Rightarrow (f(A), f(B)) \in \delta_2\).

DEFINITION 4.2 Let \((X, \delta_1, \delta_2), (Y, \delta_1', \delta_2')\) be two biquasi-proximity spaces. A function \(f : X \rightarrow Y\) is said to be \textit{pairwise proximity mapping} if \(f : (X, \delta_1) \rightarrow (Y, \delta_1')\) and \(f : (X, \delta_2) \rightarrow (Y, \delta_2')\) is a proximity mapping.

It is clear that \(f : (X, \delta_1) \rightarrow (Y, \delta_2)\) is a proximity mapping iff \(f : (X, \delta_1^{*}) \rightarrow (Y, \delta_2^{*})\) is a pairwise proximity mapping. If \(\delta_i^{*}\) is the proximity generated by \(\delta_i, i=1, 2\), then if \(f : (X, \delta_i) \rightarrow (Y, \delta_2)\) is a proximity mapping then so is \(f : (X, \delta_1^{*}) \rightarrow (Y, \delta_2^{*})\).

It is clear that every (pairwise) proximity mapping is (pairwise) continuous. For the converse we have the following result.

THEOREM 4.1 If \((X, \delta_1, \delta_1^{-1})\) and \((Y, \delta_2, \delta_2^{-1})\) are pairwise proximity spaces and \(X\) is pairwise compact, then every pairwise continuous function \(f : X \rightarrow Y\) is a pairwise proximity mapping.

PROOF. If \(A\) and \(B\) are subsets of \(X\) such that \((A, B) \in \delta_1\), then \((\delta_1^{-1}-clA) \cap (\delta_1-clB) \neq \emptyset\), by Theorem 3.10. Therefore \(f(\delta_1^{-1}-clA) \cap f(\delta_1-clB) \neq \emptyset\). Since \(f\) is pairwise continuous, \(f(\delta_1^{-1}-clA) \subseteq \delta_2^{-1}-cl(f(A))\) and \(f(\delta_1-clB) \subseteq \delta_2-cl(f(B))\).

Therefore \(\delta_2^{-1}-cl(f(A)) \cap \delta_2-cl(f(B)) \neq \emptyset\), which yields \((\delta_2^{-1}-cl(f(A)), \delta_2-cl(f(B))) \in \delta_2\) which is equivalent to \((f(A), f(B)) \in \delta_2\) as required.

Let \(\delta\) be a quasi-proximity on \(X\) and let \(\delta^{*}\) be the proximity on \(X\) generated by \(\delta\). Then[11]

\[ \mathcal{L}(\delta^{*}) = \sup \{ \mathcal{L}(\delta), \mathcal{L}(\delta^{-1}) \} \]

Therefore \((X, \delta, \delta^{-1})\) is compact iff the proximity space \((X, \delta^{*})\) is compact.

We shall now construct compactification of a pairwise proximity space \((X, \delta, \delta^{-1})\), whose construction is similar to that Smirnov compactification given in [8]. Let \(\delta^{*}\) be the proximity generated by \(\delta\) and let \(\bar{X}\) be the set of all clusters in \((X, \delta^{*})\). For \(A \subseteq X\), let \(\mathcal{A} = \{ \sigma \in \bar{X} : A \in \sigma \}\) and let \(f : X \rightarrow \bar{X}\) be defined by setting \(f(x) = \sigma_x\), the point cluster. Then (i) \(f\) is one-to-one, since \(\delta^{*}\) is separated and (ii) \(f(A) \subseteq \mathcal{A}\), since \(A \in \sigma_x\) for each \(x \in A\).

LEMMA 4.1 There exists a quasi-proximity \(\bar{\delta}\) on \(\bar{X}\) such that the pairwise proximity space \((\bar{X}, \bar{\delta}, \bar{\delta}^{-1})\) is compact.
PROOF. We define $\varrho$ on the power set of $X$ as follows: For subsets $P$, $Q$ of $X$, we let $(P, Q) \in \varrho$ iff $P \subset A$, $Q \subset B$ implies $(A, B) \in \delta$. It follows easily that $\varrho$ is a quasi-proximity on $X$. Let $\varrho^*$ be the proximity on $X$ generated by $\varrho$. Let $\varrho^{**}$ denote the proximity of Smirnov compactification [8]. We shall show that $\varrho^*$ and $\varrho^{**}$ are same. Since $\varrho^*$ is the smallest proximity finer than $\varrho$ and $\varrho^{-1}$ and the proximity $\varrho^{**}$ is finer than both $\varrho$ and $\varrho^{-1}$ therefore $\varrho^{*}$ is finer than $\varrho^*$. Conversely, assume $(P, Q) \notin \varrho^{**}$, then there are subsets $A$ and $B$ of $X$ such that $P \subset A$, $Q \subset B$ and $(A, B) \notin \delta^*$ the proximity generated by $\delta$. But $(A, B) \in \delta^*$ implies the existence of finite covers $\{A_1, \ldots, A_m\}$, $\{B_1, \ldots, B_n\}$ of $A$ and $B$ such that $(A_i, B_j) \in \delta$, $(B_j, A_i) \in \delta$ for any $i=1, \ldots, m$, $j=1, \ldots, n$. Clearly $\{A_i : i=1, \ldots, m\}$, $\{B_j : j=1, \ldots, n\}$ are finite covers of $P$ and $Q$ and for which $(A_i, B_j) \in \delta$ $(B_j, A_i) \in \varrho$ for any $i=1, \ldots, m$, $j=1, \ldots, n$. This proves that $(P, Q) \notin \varrho^{*}$, showing that $\varrho^{**}$ is coarser than $\varrho^*$. This together with the earlier made observation implies $\varrho^{**}=\varrho^*$. Since the space $(X, \varrho^*)$ is compact [8], $(X, \varrho, \varrho^{-1})$ is compact.

**Lemma 4.2** $(X, \delta, \delta^{-1})$ is pairwise proximally isomorphic to $f(X)$ with the subspace quasi-proximities $\varrho$ and $\varrho^{-1}$ and $f(X)$ is dense in $X$.

**Proof.** Since closure of $f(X)$ in $\mathcal{S}(\varrho^*)$ is $X$, $f(X)$ is dense in $X$. Now $(f(A), f(B)) \in \varrho$ iff $(C, D) \in \delta$, whenever $f(A) \subset C$, $f(B) \subset D$ iff $(C, D) \in \delta$, whenever $A \subset \delta^*-clC$, $B \subset \delta^*-clD$. But this last statement is equivalent to $(A, B) \in \delta$. So that $(f(A), f(B)) \in \varrho$ iff $(A, B) \in \delta$. Thus $X$ is pairwise proximally isomorphic to $f(X)$.

**Lemma 4.3** Every pairwise proximity mapping $g$ of $(X, \delta, \delta^{-1})$ onto a dense subset of a compact space $(Y, \delta_1, \delta_1^{-1})$ extends to a pairwise proximity isomorphism $g$ of $(X, \varrho, \varrho^{-1})$ onto $(Y, \delta_1, \delta_1^{-1})$.

**Proof.** If $\sigma$ is a cluster in $X$, there corresponds a cluster $\sigma'$ in $Y$. Since $Y$ is compact, $\sigma'$ is a point cluster. Thus every point in $Y$ determines a unique cluster (via proximity isomorphism of the dense subspace) in $X$. Thus there exists a one-to-one map $\varrho : X \rightarrow Y$, which extends $g$.

To prove that $\varrho$ is a pairwise proximity isomorphism it is sufficient to show that $\varrho : (X, \varrho) \rightarrow (Y, \delta_1)$ is a proximity isomorphism. Let $P$, $Q$ be subsets of $X$ such that if $(P, Q) \in \varrho$, then $(\varrho^{-1}-clP) \cap (\varrho-clQ) \neq \emptyset$. Hence there exists a $\sigma \in X$ such that $(P, [\sigma]) \in \varrho$ and $(\sigma, Q) \in \varrho$. Let $y = \varrho(\sigma)$, then we have $(y, \varrho(Q)) \in \delta_1$ and $(\varrho(P), [y]) \in \delta_1$ whence $(\varrho(P) \varrho(Q)), \in \delta_1$. Conversely consider $(\varrho(P)$,
\[ g(Q) \in \delta_1 \] then \((\delta_1^{-1} - \text{cl}g(P)) \cap (\delta_1^{-1} - \text{cl}g(Q)) \neq \emptyset\) since \(Y\) is compact and hence pairwise compact. Let \(y\) be in this intersection and let \(\sigma = g^{-1}(y)\). If \(A \in \sigma\) and \(P \subseteq B\), then \((g(P), A) \in \delta\) and \(g(P) \subseteq \delta_\ast\text{cl}B\), which imply \((B, A) \in \delta\) so that \((P, \{\sigma\}) \in \delta\). Similarly \((\{\sigma\}, Q) \in \delta\) from which we conclude that \((P, Q) \in \delta\).

Above three lemmas taken together prove the main results of this section.

**Theorem 4.2** Every pairwise proximity space \((X, \delta, \delta^{-1})\) is a dense subspace of a unique (up to pairwise proximity isomorphism) compact space \((X, \tilde{\delta}, \tilde{\delta}^{-1})\). If \(A, B\) are subsets of \(X\), then \((A, B) \in \delta\) iff \((\tilde{\delta}^{-1} - \text{cl}A) \cap (\tilde{\delta} - \text{cl}B) \neq \emptyset\).

**Remark** In the statement of the above theorem, we have identified \(X\) with \(f(X)\) as is usually done.

We conclude this section by an important result about the extension of maps from the spaces to their compactifications.

**Theorem 4.3** Every pairwise proximity mapping \(g\) of \((X, \delta_1, \delta_1^{-1})\) onto \((Y, \delta_2, \delta_2^{-1})\) has a unique extension \(\tilde{g}\) which is pairwise proximity mapping of the compactification of \(X\) onto the compactification of \(Y\).

**Proof.** If \(\sigma_1\) is a cluster in \(X\), then there corresponds a cluster \(\sigma_2\) in \(Y\) (since \(g: (X, \delta_1^\ast) \to (Y, \delta_2^\ast)\) is a proximity mapping) such that
\[
\sigma_2 = \{P \subseteq Y : (P, g(C)) \in \delta_2^\ast \text{ for all } C \in \sigma_1\}.
\]
Let \(\tilde{g}(\sigma_1) = \sigma_2\). Then \(\tilde{g}\) maps \(X\) to \(Y\). Clearly \(\tilde{g}(\sigma_1) = \sigma g(x)\), i.e., \(\tilde{g}\) agrees with \(g\) on \(X\) (identifying \(X\) with \(f(X)\)). To show that \(\tilde{g}\) is a proximity mapping, we shall show that \((P, Q) \in \tilde{\delta}_1\) implies \((\tilde{g}(P)), (\tilde{g}(Q)) \in \tilde{\delta}_2\), i.e., if \(g(P) \subseteq \tilde{A}\), \(g(Q) \subseteq \tilde{B}\), then \((A, B) \in \delta_2\). If \((A, B) \in \delta_2\), then there exists sets \(C\) and \(D\) in \(Y\) such that \((A, Y - C) \in \delta_2, (Y - D, B) \in \delta_2\) and \((C, D) \in \delta_2\). Since \(g(P) \subseteq \tilde{A}\), \(Y - C\) belongs to no cluster in \(\tilde{g}(P)\). For if \(Y - C\) is in some cluster in \(\tilde{g}(P)\), then \(A\) and \(Y - C\) belong to some cluster and so \((A, Y - C) \in \delta_2^\ast\), which is a contradiction. So \(g^{-1}(Y - C) = X - g^{-1}(C)\) belongs to no cluster in \(P\). This shows that \(P \subseteq g^{-1}(C)\). Similarly \(Q \subseteq g^{-1}(D)\). Since \((P, Q) \in \delta_1\), we must have \((g^{-1}(C), g^{-1}(D)) \in \delta_1\) which yields a contradiction, since \(g\) is a proximity mapping from \((X, \delta_1)\) onto \((Y, \delta_2)\). Theorefore \(g: (X, \delta_1) \to (Y, \delta_2)\) must be a proximity mapping.

That \(\tilde{g}\) is onto follows from the fact that \(f(Y) \subseteq g(X) \subseteq Y\), \(f(Y)\) is dense in \(Y\) and \(g(X)\) is compact w.r.t. \(\tilde{\delta}_1^\ast\).

We now show that \(\tilde{g}\) is unique. Suppose \(g' \neq \tilde{g}\) is another extension. Then there is a \(\sigma \in X\) such that \(g'(\sigma) \neq \tilde{g}(\sigma)\). Since \(Y\) is Hausdorff w.r.t. \(\mathcal{F}(\tilde{\delta}_2^\ast)\) and


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\( g : (X, \mathcal{T}(\mathcal{Q}_1)) \rightarrow (Y, \mathcal{T}(\mathcal{Q}_2)) \) is continuous there exists a neighbourhood \( E \) of \( \sigma \) such that \( g(E) \cap g'(E) = \emptyset \). Since \( f(X) \) is dense in \( X \), there exists \( x \in X \) such that \( \sigma_x \in E \cap f(X) \). For such a \( \sigma_x \), \( g(\sigma_x) \neq g'(\sigma_x) \). Therefore \( g \) and \( g' \) do not agree on \( f(X) \) and hence not on \( X \).

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**REFERENCES**