A RELATION SATISFIED BY SOLUTIONS OF THE ADJOINT EQUATION

By W.J. Kim

Let \( y_1, y_2, \ldots, y_n \) be \( n \) linearly independent solutions of the differential equation

\[
y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0y = 0,
\]

where \( p_i \in \mathbb{C}, \ i = 0, 1, \ldots, n-1 \), and let \( W = |y_i^{(j-1)}|_{i,j=1}^n \) be the Wronskian. It is well-known [1] that

\[
v = \frac{1}{W} \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} \\ y_1' & y_2' & \cdots & y_{n-1}' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} \end{vmatrix}
\]

is a solution of the adjoint equation

\[
v^{(n)} - (p_{n-1}v)^{(n-1)} + (p_{n-2}v)^{(n-2)} - \cdots + (-1)^{n}p_0v = 0.
\]

We shall prove the following generalization of (2).

**THEOREM 1.** If \( y_1, y_2, \ldots, y_n \) are \( n \) linearly independent solutions of (1), there exist \( n \) linearly independent solutions \( v_1, v_2, \ldots, v_n \) of (3) such that

\[
\begin{vmatrix} v_1 & v_2 & \cdots & v_k \\ v_1' & v_2' & \cdots & v_k' \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{(k-1)} & v_2^{(k-1)} & \cdots & v_k^{(k-1)} \end{vmatrix} = \frac{1}{W} \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-k} \\ y_1' & y_2' & \cdots & y_{n-k}' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-k-1)} & y_2^{(n-k-1)} & \cdots & y_{n-k}^{(n-k-1)} \end{vmatrix}
\]

For the proof of this theorem, we require a few results from the theory of determinants. Each element \( a_{ij} \) of the determinant \( D = |a_{ij}|_{i,j=1}^n \) has a cofactor \( A_{ij} \). Put \( \Delta = |A_{ij}|_{i,j=1}^n \). Then it is easily confirmed that \( Dd = D^n \). If \( D \neq 0 \), we have

\[
\Delta = D^{n-1}.
\]

If \( (n-m) \) rows and \( (n-m) \) columns in \( D \) are deleted, there results an \( m \times m \)
determinant $M = |a_{rj}|_{i,j=1}^m$. This determinant $M$ is called an \textit{mth-order minor} of $D$.

On the other hand, if we delete from $D$ the rows and columns to which the elements of $M$ belong, we get an $(n-m) \times (n-m)$ determinant $N$. $N$ is called the \textit{complement} of $M$. The \textit{algebraic complement} $\tilde{M}$ of an \textit{mth-order minor} $M$ is defined to be $(-1)^{r_1 + \cdots + r_n + \cdots + s} N$.

**Lemma 1 [3].** Let $\mathfrak{M}$ be a \textit{pth-order minor} of $D$, and $M$ the corresponding minor of $D$ (i.e., $M$ has the same row and column indices as $\mathfrak{M}$). Then

$$\mathfrak{M} = D^{p-1} \tilde{M},$$

provided $D \neq 0$.

Let $\mathfrak{S}_n$ be the set of all permutations of the integers between 1 and $n$. Then

$$D = \sum_{\pi \in \mathfrak{S}} (\text{sgn } \pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)},$$

where sgn $\pi$ is $+1$ or $-1$ according as $\pi$ is even or odd. From this representation of $D$, we easily deduce the following lemma.

**Lemma 2.**

$$D = |a_{ij}|_{i,j=1}^n = |(-1)^{i+j} a_{ij}|_{i,j=1}^n = |a_{n+1-i,n+1-j}|_{i,j=1}^n.$$

We are now ready for the proof of Theorem 1.

**Proof of Theorem 1.** Let $\sigma_{ij}$ be the minor of $y^{(i-1)}_j$ in $W$. Then $v_k = \sigma_{n+1-k}^{y^{(i-1)}} W$, $k=1, 2, \ldots, n$, are solutions of (3) [1]. To prove the linear independence, it suffices to show that the Wronskian $\mathfrak{W} = |v^{(i-1)}_j|_{i,j=1}^n$ does not vanish. Put $\mathfrak{W}_k = |v^{(i-1)}_j|_{i,j=1}^k$ and $\Sigma_k = |\sigma_{ij}|_{i,j=1}^n$. Then it is easily confirmed that

$$\mathfrak{W}_k = \frac{1}{W^k} \Sigma_{n+1-k}, \quad k=1, 2, \ldots, n.$$  

Hence,

$$\mathfrak{W} = \mathfrak{W}_n = \frac{1}{W^n} \Sigma_1 = \frac{1}{W^n} W^{n-1} = \frac{1}{W},$$

the third equality following from (5) and Lemma 2. Therefore, the Wronskian $\mathfrak{W}$ does not vanish.

By Lemmas 1 and 2, we have

$$\Sigma_{n+1-k} W^{k-1} |y^{(i-1)}_j|_{i,j=1}^n,$$

$k=1, 2, \ldots, n-1$. From (6) and (7), we conclude
\[ g_k = \frac{1}{W} |y_j^{(i-1)}|_{i,f=1}^{n-k} \]

\( k=1, 2, \ldots n-1 \), establishing (4).

We remark that (4) holds for \( k=n \) if we set \( |y_j^{(i-1)}|_{i,f=1}^0 = 1 \).

A solution \( y \) of (1) is said to have a zero of order \( k \) at \( \xi \) if \( y^{(k)}(\xi) = \cdots = y^{(k)}(\xi) = 0 \) if further \( y^{(k)}(\xi) \neq 0 \), we say that \( y \) has a zero of order exactly \( k \) at \( \xi \).

**Theorem 2.** If (1) has a nontrivial solution \( y \) with a zero of order \( k \) at \( \xi \) and a zero of order \( n-k \) at \( \zeta \), then (3) has a nontrivial solution \( v \) with a zero of order \( n-k \) at \( \xi \) and a zero of order \( k \) at \( \zeta \).

**Proof.** Let \( y_1, y_2, \ldots, y_n \) be solutions of (1) satisfying \( y_j^{(i-1)}(\xi) = 0, i, j = 1, 2, \ldots, n \). Then the \( v_k \) as defined in Theorem 1, satisfies

\[ v_k^{(i)}(\xi) = v_k^{(i-1)}(\xi) = \cdots = v_k^{(n-k)}(\xi) = 0, \quad (8) \]

\( k=1, 2, \ldots n-1 \). Since the \( y \) has a zero of order \( k \) at \( \xi \), \( y = c_1y_1 + c_2y_2 + \cdots + c_{n-k}y_{n-k} \) for some constants \( c_1, c_2, \ldots, c_{n-k} \) not all zero. Furthermore, \( |y_j^{(i-1)}(\xi)|_{i,j=1}^{n-k} \neq 0 \) because the \( y \) has a zero of order \( n-k \) at \( \zeta \). In view of (4), this implies \( |v_j^{(i-1)}(\xi)|_{i,j=1}^{n-k} = 0 \). Hence, there exists a set of constants \( C_1, C_2, \ldots, C_k \) such that \( v = C_1v_1 + \cdots + C_kv_k \) is a nontrivial solution of (3) with a zero of order \( k \) at \( \zeta \). That this \( v \) has a zero of order \( n-k \) at \( \xi \) is immediate from (8).

Sherman [5, Theorem 10] obtained a similar result under the stronger condition that \( y \) have a zero of order exactly \( k \) at \( \xi \) and a zero of order \( n-k \) at \( \zeta \).

The equation

\[ (3 \sin^2 x \cos^2 x - 2)y'' - 6 \sin x \cos x (\cos^2 x - \sin^2 x)y'' - (9 \sin^2 x \cos^2 x + 14)y' = 0, \quad (9) \]

where \( 3 \sin^2 x \cos^2 x - 2 < 0 \), has three linearly independent solutions \( \sin^2 x \cos x \), \( \cos^2 x \sin x \), and 1 [6]. The solution \( \cos^2 x \sin x \) has double zeros at \( -\pi/2 \) and \( \pi/2 \). Therefore, (9) cannot have a solution with a zero of order exactly 1 at \( -\pi/2 \) and a zero of order 2 at \( \pi/2 \). This example shows that the condition in [5, Theorem 10] is indeed stronger than that in Theorem 2.

Suppose \( p_0, p_1, \ldots, p_{n-1} \) in (1) are real-valued, continuous functions defined on an interval \( I \). For the even-order equation \( (n=2m) \), we say that (1) is
disconjugate in the sense of Reid [4] on I if none of its nontrivial solutions have two \( n \)-th order zeros on I. By Theorem 2 we see that (1) is disconjugate in the sense of Reid if and only if (3) is disconjugate in the sense of Reid. Moreover, if \( P_i \in \mathcal{C}^i(I) \), (3) may be cast into the form of (1):

\[
v^{(n)} + q_{n-1}v^{(n-1)} + \cdots + q_0v = 0,\]

where

\[
q_i = \sum_{k=i}^{n-1} (-1)^{n-k} \binom{k}{i} p_k^{(k-i)},
\]

\( i=0, 1, \ldots, n-1 \). Since (1) with \( n=2m \) is known to be disconjugate in the sense of Reid on \((-c, c)\) if

\[
\sum_{k=1}^{m} \frac{|P_{2m-k}(x)|}{k!} (c+|x|)^{k-\sum_{k=m+1}^{2m} \frac{|P_{2m-k}(x)|}{k!} (c-|x|)^{k-m}(c+|x|)^m \leq 1
\]

[2, Theorem 2.3], we have the following result.

**THEOREM 3.** Assume that \( p_i \in \mathcal{C}^i; i=0, 1, \ldots, 2m-1 \), is a real-valued function defined on \((-c, c)\). The differential equation

\[
y^{(2m)} + p_{2m-1}y^{(2m-1)} + \cdots + p_0y = 0
\]

is disconjugate in the sense of Reid on \((-c, c)\) if

\[
\sum_{k=1}^{m} \frac{|q_{2m-k}(x)|}{k!} (c+|x|)^{k-\sum_{k=m+1}^{2m} \frac{|q_{2m-k}(x)|}{k!} (c-|x|)^{k-m}(c+|x|)^m \leq 1,
\]

where \( q_0, q_1, \ldots, q_{2m-1} \) are defined as in (10) with \( n=2m \).

By using Theorem 3, the differential equation

\[
y^{(2m)} + \left[ \frac{(2m-1)!}{2(1-x^2)^{m-1}} y \right]' = 0
\]

is easily shown to be disconjugate in the sense of Reid on \((-1, 1)\). However, Theorem 2.3 in [2] is inconclusive as to the disconjugacy of (11).

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**REFERENCES**


A Relation Satisfied by Solutions of the Adjoint Equation


