POINCARÉ PERIOD RELATION ON COMPACT RIEMANN SURFACES

By Ok Young Yoon

1. Introduction.

Let $S$ be a compact Riemann surface of genus $g \geq 2$, $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_g) : \Lambda = (\delta_1, \delta_2, \ldots, \delta_g)$ (abbreviated by $(\Gamma, \Lambda)$) a canonical homology basis on $S$ and $d\mu_1, d\mu_2, \ldots, d\mu_g$ a normalized abelian differentials of the first kind on $S$, uniquely determined by the given homology basis $(\Gamma, \Lambda)$. Such differential $d\mu_i$ uniquely determined by the given homology basis $(\Gamma, \Lambda)$, such that

$$\int_{\gamma_j} d\mu_i = \delta_{ij}, \quad \int_{\delta_j} d\mu_i = \pi_{ij}, \quad i, j = 1, 2, \ldots, g.$$  

The $g \times g$ matrix $\Pi = (\pi_{ij})$ is said to be the period matrix (by contrast, $g \times 2g$ matrix $(1_g, \Pi)$, where $1_g$ is the $g \times g$ identity matrix, the full period matrix) of $S$ and $(\Gamma, \Lambda)$.

It is well known that $\Pi$ is complex symmetric with positive definite imaginary part, the set of all such matrices is generally called the Siegel (or generalized) upper-half plane $\mathbb{H}_g$ of degree (or genus) $g$, and that are holomorphic functions of $3g-3$ complex parameters, "the moduli", for a non-hyperelliptic Riemann surface, of $2g-1$ complex parameters for a hyperelliptic Riemann surface, $S$ of genus $g \geq 3$ [4]. Consequently, there are $\frac{(g-2)(g-3)}{2}$ holomorphic relations for a non-hyperelliptic Riemann surface $S$ of genus $g \geq 4$ and $\frac{(g-1)(g-2)}{2}$ holomorphic relations for a hyperelliptic Riemann surface $S$ of genus $g \geq 3$ among $\pi_{ij}$. Such holomorphic relations of $\pi_{ij}$ are usually called the period relations on compact Riemann surfaces. One of the classical problems in the theory of compact Riemann surfaces is to formulate such relations, for more than a century since Riemann.

In 1888 [5, 6], Schottky first derived a period relation for $g = 4$ and 5

$$\sqrt{\tau_1} \pm \sqrt{\tau_2} \pm \sqrt{\tau_3} = 0,$$

where $\tau_i, i = 1, 2, 3$, are the products of 8 theta constants associated with $S$ and a definite canonical homology basis $(\Gamma, \Lambda)$.

On the other hand, in 1895 [2], Poincaré obtained an approximate period relation...
(3) \[ \sqrt{\pi_{13}\pi_{14}\pi_{23}\pi_{24}} \pm \sqrt{\pi_{12}\pi_{14}\pi_{32}\pi_{34}} \pm \sqrt{\pi_{12}\pi_{13}\pi_{42}\pi_{43}} = 0 \]

for compact Riemann surfaces of genus \( g=4 \) whose period matrices \( \Pi \)'s are close to diagonal form. Recently, Rauch [3] showed that Schottky's period relation implies Poincaré's approximate period relation for compact Riemann surfaces of genus \( g=4 \), and I [7] showed that a period relation of Schottky type \( \sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0 \) holds on hyperelliptic Riemann surfaces of genus \( g \geq 4 \), which can be recognized as a generalization to a relation for genus \( g>4 \) from a relation for genus \( g=4 \).

In the present paper, on hyperelliptic Riemann surfaces of genus \( g \geq 4 \), we shall obtain a period relation of Schottky type

(4) \[ \sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0, \]

where

(5) \[
\begin{align*}
& r_1 = \theta_{1010\ldots0} [10101\ldots0] \theta_{1010\ldots0} [11100\ldots0] \theta_{11100\ldots0} [11100\ldots0] \theta_{11100\ldots0} [10101\ldots0] \times \\
& \theta_{0010\ldots0} [11001\ldots0] \theta_{10010\ldots0} [11000\ldots0] \theta_{00010\ldots0} [11000\ldots0] \theta_{00010\ldots0} [11000\ldots0]
\end{align*}
\]

\[
\begin{align*}
& r_2 = \theta_{1100\ldots0} [11000\ldots0] \theta_{1100\ldots0} [11101\ldots0] \theta_{01110\ldots0} [10110\ldots0] \theta_{01110\ldots0} [10110\ldots0] \times \\
& \theta_{1010\ldots0} [11000\ldots0] \theta_{01010\ldots0} [11000\ldots0] \theta_{00100\ldots0} [11000\ldots0] \theta_{00100\ldots0} [11000\ldots0]
\end{align*}
\]

\[
\begin{align*}
& r_3 = \theta_{1110\ldots0} [10100\ldots0] \theta_{1010\ldots0} [11110\ldots0] \theta_{01110\ldots0} [10110\ldots0] \theta_{01110\ldots0} [10110\ldots0] \times \\
& \theta_{0100\ldots0} [10100\ldots0] \theta_{01010\ldots0} [10100\ldots0] \theta_{00100\ldots0} [10100\ldots0] \theta_{00100\ldots0} [10100\ldots0]
\end{align*}
\]

and then derive a Poincaré's approximate period relation

(6) \[ \sqrt{\pi_{13}\pi_{14}\pi_{23}\pi_{24} + 0(e^{10})} \pm \sqrt{\pi_{12}\pi_{14}\pi_{32}\pi_{34} + 0(e^{10})} \pm \sqrt{\pi_{12}\pi_{13}\pi_{42}\pi_{43} + 0(e^{10})} = 0, \]

where \( \varepsilon = \max \{ |\pi_{ij}| \} \), from (4) in case the period matrices are close to diagonal form.

In Section 2 we review some of well-known facts about compact Riemann surfaces and Riemann theta functions.

In Section 3 we obtain (4) on a hyperelliptic Riemann surface \( S \) of genus \( g \geq 4 \), choosing suitably a canonical homology basis \( (\mathcal{T}, \mathcal{A}) \) and considering two kinds of multiplicative functions on \( S \).

In Section 4 we give the precise computations to derive (5) from (4) on \( S \) in case the period matrix \( \Pi \) of \( S \) and \( (\mathcal{T}, \mathcal{A}) \) is close to diagonal form.
2. Terminologies and Notations.

Throughout this paper we primarily concern for a compact Riemann surface $S$ of genus $g \geq 4$, but assume for the moment $g \geq 1$.

Let $C$ be the field of complex numbers. $2g$ columns $e^{(j)}$, $\pi^{(j)}$, $j=1,2,\ldots,g$, of the full period matrix $(1_g,1)$ of $S$ and $(I,\mathcal{A})$ are linearly independent over the reals $\mathbb{R}$ and generates a discrete abelian subgroup $L$ of $C^g$ of $g$ complex variables. The quotient group $C^g/L$ is called the Jacobi variety $J(S)$ of $S$ and $J(S)$ is a compact abelian group.

An integral linear combination of $e^{(j)}$, $\pi^{(j)}$, $j=1,2,\ldots,g$, is called a period in $J(S)$. The half of a period, $(8)$

$\frac{1}{2} \{e^{(j)}\}$

is called a half period in $J(S)$.

Fixing a point $P_0$ as a base point on $S$, we can define a map $u : S \rightarrow J(S)$ by $u(P) = (\int_{P_0}^P du_1, \ldots, \int_{P_0}^P du_g)$ for each point $P$ on $S$. Since a change of a path of integration from $P_0$ to $P$ differs by a period, $u$ has a well defined image in $J(S)$. Furthermore, this map $u$ can be extented to an arbitrary divisor $S = \sum_{j=1}^m P_j$ on $S$ by

$(9)$

$u(\zeta) = \langle u_\zeta(\zeta) \rangle = \left( \sum_{j=1}^m n_j \int_{P_0}^{P_j} du_i \right)$

where $i=1,2,\ldots,g$ and $n_j,j=1,2,\ldots,m$, are integers.

$2 \times g$ matrix

$\left[ \begin{array}{c} \mu \\ \mu' \end{array} \right] = \left[ \begin{array}{c} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_g \end{array} \right] \\
\left[ \begin{array}{c} \mu_1' \\ \mu_2' \\ \vdots \\ \mu_g' \end{array} \right]$

where $\mu$, $\mu'$, $j=1,2,\ldots,g$, are 0 or 1, i.e.,

in $Z_2$, is called a characteristic.

$\left[ \begin{array}{c} \mu \\ \mu' \end{array} \right] \begin{array}{c} \text{is called even or odd depending on whether } \sum_{j=1}^g \mu_j \mu_j' \equiv 0 \text{ or } 1 \text{ (mod. 2).} \end{array}$

There are $2^g - 1$ (or $2^g - 1$) odd characteristics.

A meromorphic multi-valued function $f$ on $S$ is called multiplicative with characteristic

$\left[ \begin{array}{c} \nu \\ \nu' \end{array} \right] = \left[ \begin{array}{c} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_g \end{array} \right] \\
\left[ \begin{array}{c} \nu_1' \\ \nu_2' \\ \vdots \\ \nu_g' \end{array} \right]$

where $\nu$, $\nu'$, $j=1,2,\ldots,g$, are 0 or 1, if a continuation of $f$ along $\gamma_j$ (or $\delta_j$) carries to $(-1)^{\nu_j} f$ (or $(-1)^{\nu'_j} f$).

For $u_1, \ldots, u_g$ in $C^g$, $T = (t_{ij})$ in $\mathfrak{g}$ and a characteristic $\left[ \begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right]$, the function
defined by

$$\theta^{[\varepsilon]}(u, T) = \sum_{N \in \mathbb{Z}^d} \exp 2\pi i \left[ \frac{1}{2}(N + \frac{\varepsilon'}{2})T(N + \frac{\varepsilon'}{2}) + (N + \frac{\varepsilon'}{2})(u + \frac{\varepsilon'}{2}) \right]$$

is called (the first order) theta function with characteristic $[\varepsilon']$ and theta matrix $T$. The theta constant with characteristic $[\varepsilon']$ at $T$ is

$$\theta^{[\varepsilon']}(0, T) = \theta^{[\varepsilon']}.\]

\(\theta^{[\varepsilon']}(u, T)\) converges absolutely and uniformly on compact subsets of \(C^d \times \mathbb{S}^e\), and hence it is an analytic function on \(C^d \times \mathbb{S}^e\). We note that the theta constant \(\theta^{[\varepsilon']}(0, T)\) is an analytic function of $T$. The following properties of theta functions will be useful in the future:

12. Functional equation: \(\theta^{[\varepsilon']}(u + \frac{\mu}{\mu'}, T) = \exp(\pi i [(\varepsilon \cdot \mu' - \varepsilon' \cdot \mu) - 2 \mu \cdot u] T) \theta^{[\varepsilon']}(u, T)\).

13. Reduction formula: \(\theta^{[\varepsilon'] + 2 \mu}(u, T) = (-1)^{\varepsilon \cdot \mu} \theta^{[\varepsilon']}(u, T)\).

14. Substitution formula: \(\theta^{[\varepsilon']}(u + \frac{\mu}{\mu'}, T) = \exp \pi i \left[ -\frac{\mu T \mu}{4} - \frac{1}{2} \mu \cdot (\varepsilon + \mu') - u \cdot u \right] \theta^{[\varepsilon'] + \mu}(u, T)\).

By (14), \(\theta^{[\varepsilon']}(u, T)\) is even or odd function of $u$ depending on whether $[\varepsilon']$ is even or odd.

Hence there are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta functions of $u$, and theta constants \(\theta^{[\varepsilon']}\) with odd characteristic $[\varepsilon']$ always vanish, but not \(\theta^{[\varepsilon']}$ with even characteristic $[\varepsilon']$ necessarily.

Since the period matrix $\Pi$ of a compact Riemann surface $S$ of genus $g \geq 1$ with a canonical homology basis $(\mathcal{T}, \mathcal{A})$ is in $\mathbb{S}^e$, we can have theta functions defined by \(\theta^{[\varepsilon']}(u(P), \Pi)\) on $S$, choosing a base point $P_0$ on $S$ and replacing $u$ by $u(P)$ and $T$ by $\Pi$ in (10). These functions are called Riemann theta functions associated with $S$ and $(\mathcal{T}, \mathcal{A})$. If \(\theta^{[\varepsilon']}(u(P), \Pi)\) does not identically vanish on $S$, it has well defined and uniquely determined $g$ zeros $\zeta = P_1 \cdots P_g$ on $S$ such that $u(\zeta) + K = [\varepsilon']$, where $K$ is a vector of Riemann constants depending on a base point $P_0$.
Poincaré Period Relation on Compact Riemann Surfaces

3. Period relation of Schottky type.

Let $S$ be a hyperelliptic Riemann surface of genus $g \geq 4$ which can be realized as the Riemann surface of an algebraic function $w$ on $S$, satisfying an irreducible algebraic equation $w^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2g-1})$, where $\lambda_j$, $j=1, 2, \ldots, 2g-1$, are mutually distinct finite different from 0 and 1.

We choose a canonical homology basis $(\Gamma, A)$ on $S$ as follows:

Now, taking a point $P_0$ (with $z(P_0) = \lambda_1$) on $S$ as a base point and finding the uniquely determined normalized abelian differentials $du_i$, $i=1, 2, \ldots, g$, with respect to the chosen $(\Gamma, A)$ on $S$, we have a map $u: S \rightarrow J(S)$.

In particular, we can find all the images of $2g+2$ branch points under $u$ in $J(S)$:

<table>
<thead>
<tr>
<th>$u(\lambda_1)$</th>
<th>(00000...0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(\lambda_2)$</td>
<td>(10000...0)</td>
</tr>
<tr>
<td>$u(\lambda_3)$</td>
<td>(10000...0)</td>
</tr>
<tr>
<td>$u(\lambda_4)$</td>
<td>(11000...0)</td>
</tr>
<tr>
<td>$u(\lambda_5)$</td>
<td>(11000...0)</td>
</tr>
<tr>
<td>$u(0)$</td>
<td>(111000...0)</td>
</tr>
</tbody>
</table>
A vector $K(\lambda_1)$ of Riemann constants with respect to a base point $P_0$ is given by

\begin{equation}
K(\lambda_1) = \begin{pmatrix}
\begin{bmatrix}
g & g-1 & g-2 & \cdots & 1
\end{bmatrix}
\end{pmatrix}
\sum_{k=1}^{g} u(\lambda_{2k-1}).
\end{equation}

At this point we can find many non-zero Riemann theta functions associated with $S$ and $(\Gamma, \Delta)$. In particular,

\begin{align*}
\theta_{\begin{bmatrix}11110\cdots0\end{bmatrix}}^{(11100\cdots0)}(u(P), \Pi) & \quad 0, \lambda_1, \lambda_2, \lambda_4, \lambda_6, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}00010\cdots0\end{bmatrix}}^{(00110\cdots0)}(u(P), \Pi) & \quad \infty, \lambda_1, \lambda_2, \lambda_4, \lambda_6, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}10000\cdots0\end{bmatrix}}^{(10000\cdots0)}(u(P), \Pi) & \quad 0, \lambda_1, \lambda_2, \lambda_6, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}00100\cdots0\end{bmatrix}}^{(01100\cdots0)}(u(P), \Pi) & \quad \infty, \lambda_1, \lambda_2, \lambda_5, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}10000\cdots0\end{bmatrix}}^{(11100\cdots0)}(u(P), \Pi) & \quad 0, \lambda_1, \lambda_2, \lambda_5, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}01100\cdots0\end{bmatrix}}^{(11000\cdots0)}(u(P), \Pi) & \quad \infty, \lambda_1, \lambda_2, \lambda_5, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}10110\cdots0\end{bmatrix}}^{(10110\cdots0)}(u(P), \Pi) & \quad 0, \lambda_1, \lambda_2, \lambda_5, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}01010\cdots0\end{bmatrix}}^{(10010\cdots0)}(u(P), \Pi) & \quad \infty, \lambda_1, \lambda_2, \lambda_5, \lambda_{10}, \cdots, \lambda_{2g-2}, \\
\theta_{\begin{bmatrix}11110\cdots0\end{bmatrix}}^{(10010\cdots0)}(u(P), \Pi) & \quad 1, \lambda_1, \lambda_2, \lambda_4, \lambda_{10}, \cdots, \lambda_{2g-2}.
\end{align*}
\[ \theta_{[0001\ldots0]}(u(P), \Pi) \]
\[ \infty, \lambda_1, \lambda_2, \lambda_4, \lambda_8, \ldots, \lambda_{2g-2}, \]
\[ \theta_{[1100\ldots0]}(u(P), \Pi) \]
\[ 1, \lambda_1, \lambda_2, \lambda_5, \lambda_8, \lambda_{10}, \ldots, \lambda_{2g-2}, \]
\[ \theta_{[0110\ldots0]}(u(P), \Pi) \]
\[ \infty, \lambda_1, \lambda_2, \lambda_5, \lambda_8, \lambda_{10}, \ldots, \lambda_{2g-2}, \]
\[ \theta_{[1000\ldots0]}(u(P), \Pi) \]
\[ 1, \lambda_1, \lambda_5, \lambda_8, \lambda_{10}, \ldots, \lambda_{2g-2}, \]
\[ \theta_{[0110\ldots0]}(u(P), \Pi) \]
\[ \infty, \lambda_1, \lambda_5, \lambda_8, \lambda_{10}, \ldots, \lambda_{2g-2}, \]
\[ \theta_{[1110\ldots0]}(u(P), \Pi) \]
\[ 1, \lambda_1, \lambda_5, \lambda_8, \lambda_{10}, \ldots, \lambda_{2g-2}, \]
\[ \theta_{[0001\ldots0]}(u(P), \Pi) \]
\[ \infty, \lambda_1, \lambda_2, \lambda_5, \lambda_8, \lambda_{10}, \ldots, \lambda_{2g-2}, \]
\[ \frac{\theta_{[1110\ldots0]}(u(P), \Pi)}{\theta_{[0001\ldots0]}(u(P), \Pi)} \]
\[ \text{is multiplicative function on } S \text{ with characteristic } \begin{bmatrix} 1110\ldots0 \\ 0010\ldots0 \end{bmatrix}, \text{ with a zero 0 and a pole } \infty. \]

On the other hand, we easily see that \( \sqrt{z} \) is also multiplicative function on \( S \) with characteristic \( \begin{bmatrix} 1110\ldots0 \\ 0010\ldots0 \end{bmatrix} \), with a zero 0 and a pole \( \infty \). Hence there exists a constant \( C \) such that

\[ (19) \quad \sqrt{z} = C \frac{\theta_{[1110\ldots0]}(u(P), \Pi)}{\theta_{[0001\ldots0]}(u(P), \Pi)}. \]

Putting \( P \) with \( z(P) = 1 \),

\[ (20) \quad C = \frac{\theta_{[0001\ldots0]}(u(P), \Pi)}{\theta_{[0011\ldots0]}(u(P), \Pi)} \]
\[ = -i \frac{\theta_{[1110\ldots0]}(u(P), \Pi)}{\theta_{[0001\ldots0]}(u(P), \Pi)} \]
\[ = -i \frac{\theta_{[1110\ldots0]}(u(P), \Pi)}{\theta_{[1000\ldots0]}(u(P), \Pi)} \]

using (13) and (15). Consequently, we have from (19) and (20)

\[ (21) \quad \sqrt{\lambda_3} = -i \frac{\theta_{[1110\ldots0]}(u(P), \Pi)}{\theta_{[0001\ldots0]}(u(P), \Pi)} \frac{\theta_{[0110\ldots0]}(u(P), \Pi)}{\theta_{[1110\ldots0]}(u(P), \Pi)} \]

again using (13), (15) and putting \( P \) with \( z(P) = \lambda_3 \) in (19).

Carrying out the similar computations for the rest of functions listed in (18), we obtain the following four different expressions of \( \sqrt{\lambda_3} \) and \( \sqrt{\lambda_3 - 1} \), respecti-
To obtain (23), we only need to consider two kinds of multiplicative functions on $S$ with characteristic $\begin{bmatrix} 11110 \cdots 0 \\ 00100 \cdots 0 \end{bmatrix}$, with a zero 1 and a pole $\infty$, for example $\theta \begin{bmatrix} 11110 \cdots 0 \\ 00100 \cdots 0 \end{bmatrix}$ and $\theta \begin{bmatrix} 00100 \cdots 0 \\ 00100 \cdots 0 \end{bmatrix}$, etc. We note that denominators in (22) and (23) are coincided.

Since for any complex number $\lambda \neq 0$, $\lambda - (\lambda - 1) = 1$ and hence

$$
\sqrt[\lambda]{\lambda_3} = \sqrt[\lambda]{\lambda_3} \cdot \sqrt[\lambda]{\lambda_3} \cdot \sqrt[\lambda]{\lambda_3} = \sqrt[\lambda]{\lambda_3 - 1} \cdot \sqrt[\lambda]{\lambda_3 - 1} \cdot \sqrt[\lambda]{\lambda_3 - 1} = \pm 1.
$$

Putting (22) and (23) in (24), and then rearranging the terms suitably, we finally obtain a period relation of Schottky type on a hyperelliptic Riemann surface $S$ of genus $g \geq 4$.

**THEOREM 1.** On a hyperelliptic Riemann surface $S$ of genus $g \geq 4$ with a canonical homology basis $(\Gamma, \Delta)$ as shown in Figure, a period relation of Schottky type

$$
\sqrt[\lambda]{\tau_1} \pm \sqrt[\lambda]{\tau_2} \pm \sqrt[\lambda]{\tau_3} = 0
$$

holds, where $\tau_k$, $k = 1, 2, 3$, are given by (5).

4. Poincare's approximate period relation.

As in Section 3, we consider a hyperelliptic Riemann surface $S$ of genus $g \geq 4$ endowed with a canonical homology basis $(\Gamma, \Delta)$ as shown in Figure. We assume further that the period matrix II of $S$ and $(\Gamma, \Delta)$ is very close to diagonal form

$$
\Pi = \begin{bmatrix}
\pi_{11} & 0 \\
0 & \ddots \\
& \pi_{22} & 0 \\
& & \ddots & \ddots \\
& & & \ddots & \pi_{gg}
\end{bmatrix}
$$


Since each Riemann theta constant appeared in a period relation of Schottky type in Theorem 1 is an analytic function of entires \( \pi_{ij} \)'s, and hence as of the \( g(g-1)/2 \) off-diagonal entries \( \pi_{ij} \)'s, \( 1 \leq i < j \leq g \), of \( \Pi \).

In case \( \Pi \) is very close to \( \Pi \), we can expand each theta constant shown in (5) in a Maclaurin series about "the origin" \( \pi_{ij} = 0 \), \( 1 \leq i < j \leq g \).

In fact, since

\[
\theta[e_1^{\ldots e_g}](\Pi) = \prod_{k=1}^{g} \theta[e_k^{\ldots e_g}](\Pi) = \prod_{k=1}^{g} \theta[e_k^{\ldots e_k}](\Pi).
\]

(25) \[ \frac{\partial}{\partial \pi_{ij}} \theta[e_1^{\ldots e_g}](\Pi) = \prod_{k=1}^{g} \theta[e_k^{\ldots e_k}](\Pi) \frac{d}{du} \theta[e_i^{\ldots e_j}](u, \pi_{ij}) \bigg|_{u=0} \]

\[
\frac{d}{du} \theta[e_j^{\ldots e_j}](u, \pi_{ij}) \bigg|_{u=0}.
\]

(26) \[ \theta[e_1^{\ldots e_g}](\Pi) = \prod_{k=1}^{g} \theta[e_k^{\ldots e_k}](\Pi) + \sum_{1 \leq i < j \leq g} \left( \frac{\partial}{\partial \pi_{ij}} \theta[e_1^{\ldots e_g}](\Pi), \pi_{ij} \right) + O(e^4)
\]

(27) \[ \theta[e_1^{\ldots e_g}](\Pi) = \prod_{k=1}^{g} \theta[e_k^{\ldots e_k}](\Pi) + \frac{1}{2\pi i} \sum_{1 \leq i < j \leq g} \left( \prod_{k=1}^{g} \theta[e_k^{\ldots e_k}], \theta'[e_i^{\ldots e_j}], \theta'[e_j^{\ldots e_j}], \pi_{ij} \right)
\]

where \( \varepsilon = \max_{1 \leq i < j \leq g} \{|\pi_{ij}|\} \) and \( \theta[e_k^{\ldots e_k}] = \frac{d}{du} \theta[e_k^{\ldots e_k}](u, \pi_{kk}) \bigg|_{u=0} \), \( 1 \leq k \leq g \).

Now, appealing to (27) and

\[ \theta'(\pi)(\omega, \tau) = \frac{d}{du} \theta(\pi)(u, \tau) \bigg|_{u=0} = -\pi \theta(\pi) \theta(\omega) \theta(\nu) \neq 0 \]

for any \( \tau \) in \( \mathcal{S} \), we can easily check that from (5)

\[
\theta^{[0]}_{[10100^{\ldots 0}]} = \pi \prod_{k=1,3}^{g} \theta(\tau)^{[0]}_{[0]} \theta(\tau)^{[1]}_{[0]} \theta(\tau)^{[0]}_{[1]} \theta(\tau)^{[0]}_{[2]} \theta(\tau)^{[0]}_{[4]},
\]

(29) \[ \theta^{[1]}_{[10100^{\ldots 0}]} = \pi \prod_{k=1,4}^{g} \theta(\tau)^{[0]}_{[0]} \theta(\tau)^{[1]}_{[0]} \theta(\tau)^{[1]}_{[1]} \theta(\tau)^{[0]}_{[3]} \theta(\tau)^{[0]}_{[1]} \theta(\tau)^{[0]}_{[4]},
\]

\[ \theta^{[1]}_{[01001^{\ldots 0}]} = \pi \prod_{k=2,3}^{g} \theta(\tau)^{[0]}_{[0]} \theta(\tau)^{[1]}_{[0]} \theta(\tau)^{[0]}_{[1]} \theta(\tau)^{[0]}_{[2]} \theta(\tau)^{[0]}_{[4]}.
\]
\[
\theta[00100\cdots 0] = \theta[0][1] \theta[0][2] \theta[1][0] \theta[0][4] \cdot \prod_{k=5}^{\infty} \theta[0][k] + O(\epsilon^4),
\]
\[
\theta[11100\cdots 0] = \theta[0][1] \theta[0][2] \theta[1][0] \theta[0][4] \cdot \prod_{k=5}^{\infty} \theta[0][k] + O(\epsilon^4),
\]
\[
\theta[00010\cdots 0] = \theta[0][1] \theta[0][2] \theta[1][0] \theta[0][4] \cdot \prod_{k=5}^{\infty} \theta[0][k] + O(\epsilon^4).
\]

Note that \( \theta[1][k] = \theta[0][k] = \theta'[1][k] = \theta'[0][k] = 0 \), \( 1 \leq k \leq g \). Multiplying together, we obtain

\[
(30) \quad r_1 = 4 \prod_{k=1}^{\infty} \left( \theta[0][0] \theta[1][0] \theta[0][4] \cdot \prod_{k=5}^{\infty} \theta[0][k] \right)^4 \cdot \prod_{k=5}^{\infty} \theta[0][k] \cdot \pi_{13} \pi_{14} \pi_{23} \pi_{24} + O(\epsilon^\infty).
\]

For \( r_2 \) and \( r_3 \), the similar computations give the expressions

\[
(31) \quad r_2 = -4 \prod_{k=1}^{\infty} \left( \theta[0][0] \theta[1][0] \theta[0][4] \cdot \prod_{k=5}^{\infty} \theta[0][k] \right)^4 \cdot \prod_{k=5}^{\infty} \theta[0][k] \cdot \pi_{12} \pi_{14} \pi_{32} \pi_{34} + O(\epsilon^\infty),
\]
\[
(32) \quad r_3 = -4 \prod_{k=1}^{\infty} \left( \theta[0][0] \theta[1][0] \theta[0][4] \cdot \prod_{k=5}^{\infty} \theta[0][k] \right)^4 \cdot \prod_{k=5}^{\infty} \theta[0][k] \cdot \pi_{12} \pi_{13} \pi_{42} \pi_{43} + O(\epsilon^\infty).
\]

By Theorem 1, we know that on a hyperelliptic Riemann surface \( S \) of genus \( g \geq 4 \),

\[
\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = A(\sqrt{\pi_{13} \pi_{14} \pi_{23} \pi_{24} + O(\epsilon^\infty)}
\pm \sqrt{\pi_{12} \pi_{14} \pi_{32} \pi_{34} + O(\epsilon^\infty)}
\pm \sqrt{\pi_{12} \pi_{13} \pi_{42} \pi_{43} + O(\epsilon^\infty)})
\]

\[
= 0
\]

and \( A = \frac{\pi}{4} \prod_{k=1}^{\infty} \left( \theta[0][0] \theta[1][0] \theta[0][4] \cdot \prod_{k=5}^{\infty} \theta[0][k] \right)^4 \neq 0 \),

and thus we have a relation (6) among \( \pi_{ij} \) in \( \Pi \).

**THEOREM 2.** On a hyperelliptic Riemann surface \( S \) of genus \( g \geq 4 \) with a canonical homology basis \( (\Gamma', \Delta) \) as shown in Figure such that the period matrix \( \Pi \) of \( S \) and \( (\Gamma', \Delta) \) is close to diagonal form, a Poincaré's approximate period relation (6) exists.

We remark that our result of Theorem 2 of course includes the result done by Rauch [3] for genus \( g = 4 \).
Poincaré Period Relation on Compact Riemann Surfaces

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REFERENCES