A NOTE ON $\mathcal{R}$-SPACES

By M.K. Singal and Pushpa Jain

In [10] O'Meara has introduced a new class of topological spaces, called $\mathcal{R}$-spaces. Following him, a regular $T_1$ space with a $\sigma$-locally finite $k$-network is called an $\mathcal{R}$-space. A collection $\mathcal{F}$ of subsets of $X$ is said to be a $k$-network for $X$ if for each compact subset $K$ of $X$ and each open subset $U$ of $X$ containing $K$ there exists a finite union $R$ of members of $\mathcal{F}$ such that $K \subset R \subset U$. $\mathcal{F}$ is said to be a pseudobase if for each compact subset $K$ of $X$ and each open subset $U$ of $X$ containing $K$ there is a $B \in \mathcal{F}$ such that $K \subset B \subset U$. $\mathcal{F}$ is said to be a network for $X$ if for each $x \in X$ and each open subset $U$ of $X$ containing $x$, there is a $B \in \mathcal{F}$ such that $x \in B \subset U$. A space $X$ with a countable pseudo-base is called an $\mathcal{R}_0$-space by Michael [7], whereas a space $X$ with a closed $\sigma$-locally finite network is called a $\sigma$-space by Okuyama [13]. The class of all $\sigma$-spaces contains the class of all $\mathcal{R}_0$-spaces, and all subparacompact spaces (that is, spaces with the property that every open covering has a $\sigma$-discrete closed refinement).

In [10,11] properties of $\mathcal{R}$-spaces parallel to $\mathcal{R}_0$-spaces have been obtained. In the present note some sum theorems for $\mathcal{R}$-spaces have been given. It is also proved that the image of an $\mathcal{R}$-space under a perfect mapping is an $\mathcal{R}$-space.

In the end, we obtain a sufficient condition for an invertible space to be an $\mathcal{R}$-space. Simple extension due to Levine [12] has also been considered for $\mathcal{R}$-spaces.

We shall first prove the locally finite sum theorem for $\mathcal{R}$-spaces which states the following:

**Theorem 1.** If $\{F_\alpha : \alpha \in \Omega\}$ is a locally finite closed covering of $X$ such that each $F_\alpha$ is an $\mathcal{R}$-space, then $X$ is an $\mathcal{R}$-space.

**Proof.** Since each $F_\alpha$ is a regular $T_1$ space, therefore $X$ is a regular $T_1$ space. Thus we shall only prove that $X$ has a $\sigma$-locally finite $k$-network if each $F_\alpha$...
has a $\sigma$-locally finite $k$-network. For each $\alpha \in \Omega$ let $\mathcal{Y}_{\alpha} = \bigcup_{n=1}^{\infty} \mathcal{Y}_{\alpha}^{n}$ be a $\sigma$-locally finite $k$-network for $F_{\alpha}$, where each $\mathcal{Y}_{\alpha}^{n}$ is locally finite in $F_{\alpha}$ (and hence in $X$). Then $\mathcal{Y} = \bigcup_{n=1}^{\infty} \mathcal{Y}_{n}$, where $\mathcal{Y}_{n} = \bigcup_{\alpha \in \Omega} \mathcal{Y}_{\alpha}^{n}$ is a $\sigma$-locally finite $k$-network for $X$.

For, let $K$ be a compact subset of $X$ and $U$ an open subset of $X$ such that $K \subseteq U$. Since every locally finite family is compact finite (that is, every compact subset intersects at most finitely many members of the family), therefore $K$ intersects at most finitely many $F_{\alpha}$'s say $F_{\alpha_{1}}, F_{\alpha_{2}}, \ldots, F_{\alpha_{k}}$. Thus $K = \bigcup_{i=1}^{k} (K \cap F_{\alpha_{i}}) \subseteq U$. For each $i = 1, 2, \ldots, k$, $K \cap F_{\alpha_{i}}$ is a compact subset of $F_{\alpha_{i}}$ which is contained in an open subset $U \cap F_{\alpha_{i}}$ of $F_{\alpha_{i}}$. Let $R_{i}$ be a finite union of members of $\mathcal{Y}_{\alpha_{i}}$ such that $K \cap F_{\alpha_{i}} \subseteq R_{i} \subseteq U \cap F_{\alpha_{i}} \subseteq U$. Therefore, $\bigcup_{i=1}^{k} R_{i}$ is a finite union of members of $\mathcal{Y}$ such that $K \subseteq \bigcup_{i=1}^{k} R_{i} \subseteq U$. Hence $\mathcal{Y}$ is a $\sigma$-locally finite $k$-network.

**COROLLARY 1.** Every disjoint topological sum of $\mathbb{R}$-spaces is an $\mathbb{R}$-space.

It has been proved by Hodel [3] that for any topological property $P$ which is closed hereditary (that is a property, which when possessed by a space, is possessed by every closed subset of it) and which satisfies the locally finite sum theorem, the following theorems are true.

**THEOREM 2.** If $\mathcal{Y}$ is a $\sigma$-locally finite open covering of a space $X$ such that the closure of each member of $\mathcal{Y}$ has the property $P$, then $X$ has the property $P$.

**THEOREM 3.** Let $X$ be a regular topological space and let $\mathcal{Y}$ be a $\sigma$-locally finite open covering of $X$ such that each member of $\mathcal{Y}$ has the property $P$ and the frontier of each member of $\mathcal{Y}$ is compact. Then $X$ has the property $P$.

**THEOREM 4.** If $\mathcal{Y}$ is a $\sigma$-locally finite elementary covering of $X$ such that each member of $\mathcal{Y}$ has the property $P$, then $X$ has the property $P$. (For the definition of elementary covering see definition 1).

**DEFINITION 1.** [Hodel, 3]. A subset $A$ of $X$ is said to be **elementary** if it is open and if there exists a sequence $\{A_{i}\}_{i=1}^{\infty}$ of open subsets of $X$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_{i}$ and $\overline{A}_{i} \subseteq A$ for all $i$. A covering consisting of elementary sets is said to be an **elementary covering**.
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DEFINITION 2. [Y. Katuta, 4]. A family \( \{A_{\alpha} : \alpha \in \Omega\} \) of subsets of \( X \) is said to be order locally finite if there is a linear ordering \( \prec \) of the index set \( \Omega \) such that for each \( \alpha \in \Omega \), the family \( \{A_{\beta} : \beta \prec \alpha\} \) is locally finite at each point of \( A_{\alpha} \).

Every \( \sigma \)-locally finite family is order locally finite, but not conversely.

In [13], Singal and Arya have proved some sum theorems for order locally finite open coverings of \( X \). Let \( P \) be a topological property which is closed hereditary and which satisfies the locally finite sum theorem, then the following two theorems hold.

THEOREM 5. Let \( \mathcal{V} \) be an order locally finite open covering of \( X \) such that the closure of each member of \( \mathcal{V} \) possesses the property \( P \). Then \( X \) possesses \( P \).

THEOREM 6. If \( \mathcal{V} \) is an order locally finite open covering of a regular space \( X \) such that each member of \( \mathcal{V} \) possesses the property \( P \) and the frontier of each member of \( \mathcal{V} \) is compact, then \( X \) has the property \( P \).

Obviously Theorems 2 and 3 of Hodel follow as corollaries to Theorems 5 and 6, respectively.

Since the property of being an \( \mathbb{R} \)-space is hereditary, therefore, in view of Theorem 1, we have the following theorems.

THEOREM 7. If \( \mathcal{V} \) is a \( \sigma \)-locally finite elementary covering of \( X \) such that each \( V \in \mathcal{V} \) is an \( \mathbb{R} \)-space, then \( X \) is an \( \mathbb{R} \)-space.

THEOREM 8. If \( \mathcal{V} \) is an order locally finite open covering of \( X \) such that the closure of each member of \( \mathcal{V} \) is an \( \mathbb{R} \)-space, then \( X \) is an \( \mathbb{R} \)-space.

THEOREM 9. If \( \mathcal{V} \) is an order locally finite open covering of a regular space \( X \) such that each member of \( \mathcal{V} \) is an \( \mathbb{R} \)-space and frontier of each member of \( \mathcal{V} \) is compact, then \( X \) is an \( \mathbb{R} \)-space.

As a consequence of the locally finite sum theorem and the closed hereditary character of \( \mathbb{R} \)-spaces, we deduce the following interesting results.

THEOREM 10. Let \( \mathcal{V} \) be a locally finite open covering of a regular space \( X \) such that each member of \( \mathcal{V} \) is an \( \mathbb{R} \)-space and frontier of each member of \( \mathcal{V} \) is Lindelöf. Then \( X \) is an \( \mathbb{R} \)-space.

PROOF. Let \( \mathcal{V} = \{V_{\alpha} : \alpha \in \Omega\} \) be the given locally finite open covering of \( X \). For each \( \alpha \in \Omega \), \( Fr V_{\alpha} \) is Lindelöf. Therefore there exists a countable subfamily \( \{V_{\alpha_i} : i=1,2,\ldots\} \) of \( \mathcal{V} \) which covers \( Fr V_{\alpha} \). Let \( F_1 = Fr V_{\alpha} \cup \bigcup_{i=2}^{\infty} V_{\alpha_i} \). Then \( F_1 \) is a.
closed subset of $\text{Fr } V_\alpha$ (and hence of $X$) such that $F_1 \subset V_\alpha$. Since $F_1$ is Lindelöf and $X$ is regular there exists an open set $U_1$ such that $F_1 \subset U_1 \subset \bar{U}_1 \subset V_\alpha$. $\bar{U}_1$, being a closed subset of an $\mathfrak{S}$-space $V_\alpha$, is an $\mathfrak{S}$-space. Suppose for each $i=1, 2, \ldots, n-1$, there exists an open set $U_i$ such that $F_i \subset U_i \subset \bar{U}_i \subset V_\alpha$, where

$$F_i=\text{Fr } V_\alpha \sim \left( \bigcup_{k=1}^{i-1} U_k \bigcup \bigcup_{k=i+1}^\infty V_\alpha \right)$$

and $\bar{U}_i$ is an $\mathfrak{S}$-space. Now let,

$$F_n=\text{Fr } V_\alpha \sim \left( \bigcup_{k=1}^{n-1} U_k \bigcup \bigcup_{k=n+1}^\infty V_\alpha \right).$$

Then $F_n$ is a closed Lindelöf subset such that $F_n \subset V_\alpha$. Again, by regularity of $X$ there exists an open set $U_n$ such that $F_n \subset U_n \subset \bar{U}_n \subset V_\alpha$, where $\bar{U}_n$ is an $\mathfrak{S}$-space. Thus by induction we obtain a family $Z'=\{U_n: n \in \mathbb{N}\}$ of open sets satisfying:

(a) $Z'$ is a covering of $\text{Fr } V_\alpha$.

(b) $\{\bar{U}_n: n \in \mathbb{N}\}$ is locally finite.

Let $F_0=V_\alpha \sim \bigcup_{k=1}^\infty U_k$; then $\{\bar{U}_n: n \in \mathbb{N}\} \cup \{F_0\}$ is a locally finite closed covering of $V_\alpha$ each member of which is an $\mathfrak{S}$-space. Hence by Theorem 1, $V_\alpha$ is an $\mathfrak{S}$-space. Thus $\{\bar{V}_\alpha: \alpha \in \Omega\}$ is a locally finite closed covering of $X$ each member of which is an $\mathfrak{S}$-space. Hence $X$ is an $\mathfrak{S}$-space, in view of Theorem 1. For details of the proof, see [14].

THEOREM 11. If $\mathcal{Y}$ be a locally finite open covering of a normal space $X$ such that each $V \in \mathcal{Y}$ is an $\mathfrak{S}$-space, then $X$ is an $\mathfrak{S}$-space.

PROOF. Let $\mathcal{Y} = \{V_\alpha: \alpha \in \Omega\}$. Since $X$ is normal, there exists an open covering $\{U_\alpha: \alpha \in \Omega\}$ of $X$ such that $\bar{U}_\alpha \subset V_\alpha$. Then $\{\bar{U}_\alpha: \alpha \in \Omega\}$ is a locally finite closed covering of $X$ such that each $\bar{U}_\alpha$ is an $\mathfrak{S}$-space. Hence $X$ is an $\mathfrak{S}$-space.

An open covering $\mathcal{Y}$ of $X$ is said to be a normal open covering if there is a sequence $\{\mathcal{Y}_n\}$ of open coverings of $X$ such that each $\mathcal{Y}_n$ is a star-refinement of $\mathcal{Y}_{n-1}$ and $\mathcal{Y}_1$ is a refinement of $\mathcal{Y}$.

THEOREM 12. Let $\mathcal{Y}$ be a normal open covering of a normal space $X$. Then $X$ is an $\mathfrak{S}$-space if each $V \in \mathcal{Y}$ is an $\mathfrak{S}$-space.

PROOF. Since $\mathcal{Y}$ is a normal open covering of the normal space $X$, therefore $\mathcal{Y}$ admits of a locally finite open refinement [8, Theorem 1.2]. Hence the
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result follows in view of Theorem 11.

THEOREM 13. If $\mathscr{V}$ be a point finite open covering of a collectionwise normal space such that each member of $\mathscr{V}$ is an $\mathfrak{K}$-space, then $X$ is an $\mathfrak{K}$-space.

PROOF. The result follows in view of Theorem 11 and the fact that in a collectionwise normal space, every point finite open covering has a locally finite open refinement [6].

THEOREM 14. Let $\mathscr{V}$ be a $\sigma$-locally finite open covering of a normal space $X$ such that each $V \in \mathscr{V}$ is an $F_\sigma$-subset of $X$. If each $V \in \mathscr{V}$ is an $\mathfrak{K}$-space, then $X$ is an $\mathfrak{K}$-space.

PROOF. By Theorem 1.2 in [8], $\mathscr{V}$ is a normal covering. Hence the result follows in view of Theorem 12.

THEOREM 15. Let $\mathscr{V}$ be a $\sigma$-locally finite open covering of a countably paracompact normal space $X$ such that each member of $\mathscr{V}$ is an $\mathfrak{K}$-space. Then $X$ is an $\mathfrak{K}$-space.

PROOF. Since every $\sigma$-locally finite open covering of a countably paracompact normal space is normal (see [9]), the result follows in view of Theorem 12 above.

THEOREM 16. Let $X$ be a regular space which is the union of two sets $A$ and $B$ such that $A$ is nonempty, compact and $B$ is paracompact. If $\mathscr{V}$ be an open covering of $X$ such that each $V \in \mathscr{V}$ is an $\mathfrak{K}$-space, then $X$ is an $\mathfrak{K}$-space.

PROOF. Let $\mathscr{V} = \{V_\alpha: \alpha \in \Omega\}$. For each $x \in A$ there is an $\alpha_x \in \Omega$ such that $x \in V_{\alpha_x}$. Since $X$ is regular, let $U_{\alpha_x}$ be an open subset of $X$ such that $x \in U_{\alpha_x} \subset \overline{U_{\alpha_x}} \subset V_{\alpha_x}$. Let $\{U_{\alpha_x_1}, \ldots, U_{\alpha_x_n}\}$ be a finite subfamily of $\{U_{\alpha_x}: x \in A\}$ such that $A \subset \bigcup_{i=1}^n U_{\alpha_x_i}$. Also, each $\overline{U_{\alpha_x_i}}$, being a subset of $V_{\alpha_x_i}$, is an $\mathfrak{K}$-space. Let $F = X \sim \bigcup_{i=1}^n U_{\alpha_x_i}$. Then $F$ is a closed subset of $X$ which is contained in $B$ and hence $F$ is a regular paracompact space. Therefore the covering $\{F \cap V_\alpha: \alpha \in \Omega\}$ of $F$ has a locally finite (in $F$ and hence in $X$) closed (in $F$ and hence in $X$) refinement $\mathscr{V}$. The covering $\mathscr{W} = \{U: U \in \mathscr{Z}\} \cup \{\overline{U}_{\alpha_x_i}: i = 1, 2, \ldots, n\}$ is then a locally finite closed covering of $X$ such that each $W \in \mathscr{W}$ is an $\mathfrak{K}$-space. Hence $X$ is an $\mathfrak{K}$-space in view of Theorem 1.

THEOREM 17. Let $X$ be a collectionwise normal space and let $X$ be the union of two sets $A$ and $B$ such that $A$ is paracompact and closed and $B$ is paracompact. If $\mathscr{V}$ is an open covering of $X$ such that each $V \in \mathscr{V}$ is an $\mathfrak{K}$-space then $X$ is an
**Theorem 18.** Let $\mathcal{F} = \{V_\alpha : \alpha \in \Omega\}$. Since $A$ is paracompact, the open covering $\{A \cap V_\alpha : \alpha \in \Omega\}$ of $A$ has a locally finite open refinement $\{U_\beta : \beta \in \Gamma\}$. Since $X$ is collectionwise normal, therefore by Lemma 1 in [1] there exists a locally finite open covering $\{W_\beta : \beta \in \Gamma\}$ of $X$ such that $A \cap W_\beta \subseteq U_\beta$ for each $\beta \in \Gamma$. For each $\alpha \in \Omega$, let $\alpha(\beta) \in \Omega$ such that $U_\beta \subseteq A \cap V_{\alpha(\beta)}$. Let $G_\beta = W_\beta \cap V_{\alpha(\beta)}$. Then $\mathcal{F} = \{G_\beta : \beta \in \Gamma\}$ is a locally finite open collection which covers $A$. Since $X$ is regular, there exists a locally finite open collection $\mathcal{F}' = \{H_\delta : \delta \in \Delta\}$ which covers $A$ and is such that each $H_\delta$ is contained in some $G_\beta$. Let $F = X - \cup \{H_\delta : \delta \in \Delta\}$. Then $F$ is a closed subset of $X$ which is contained in $B$, and hence is paracompact. As above, we obtain a locally finite closed collection $\mathcal{H}'' = \{H_\delta : \delta \in \Delta\} \cup \mathcal{H}''$ which covers $F$ such that each member of $\mathcal{H}''$ is an $\mathcal{R}$-space. Thus $\mathcal{F} = \{H_\delta : \delta \in \Delta\} \cup \mathcal{H}''$ is a locally finite closed covering of $X$ such that each member of $\mathcal{F}$ is an $\mathcal{R}$-space. Hence $X$ is an $\mathcal{R}$-space.

**Theorem 19.** Every space which contains a proper, nonempty regularly closed subset is an $\mathcal{R}$-space if and only if every regularly closed subset of $X$ is an $\mathcal{R}$-space.

**Proof.** The 'only if' part is obvious. We shall, therefore, prove the 'if' part. Let $X$ be a space containing a proper nonempty regularly closed set $U$. Therefore $U = U^{-}$. Let $U^{-} = V$. Then $V$ is contained in $U$ and so $\overline{V}$ is a proper regularly closed subset of $X$ where $V$ is regularly open. Thus $X = \overline{V} \cup (X - V)$, where $V$ and $X - V$ are both $\mathcal{R}$-spaces. Hence $X$ is an $\mathcal{R}$-space.

**Corollary 2.** A weakly regular space $X$ is an $\mathcal{R}$-space if and only if every proper regularly closed subset of $X$ is an $\mathcal{R}$-space.

**Corollary 3.** A semi-regular space $X$ is an $\mathcal{R}$-space if and only if every proper regularly closed subset of $X$ is an $\mathcal{R}$-space.
A mapping \( f : X \rightarrow Y \) is called a perfect mapping if it is closed, continuous and such that \( f^{-1}(y) \) is compact for each \( y \in Y \).

**THEOREM 22.** Let \( f : X \rightarrow Y \) be a perfect mapping. Then \( Y \) is an \( \aleph \)-space if \( X \) is so.

**PROOF.** Let \( X \) be an \( \aleph \)-space and let \( \mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n \) be a \( \sigma \)-locally finite \( k \)-network for \( X \). We shall prove that \( \mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n \), where \( \mathcal{W}_n = \{ f(V) : V \in \mathcal{W}_n \} \) is a \( \sigma \)-locally finite \( k \)-network for \( Y \). Since \( f \) is continuous, each \( \mathcal{W}_n \) will be locally finite in \( Y \).

To prove that \( \mathcal{W} \) is a \( k \)-network for \( Y \), let \( K \) be a compact subset of \( Y \) and \( U \) be an open subset of \( Y \) such that \( K \subseteq U \). Since \( f \) is a closed continuous mapping with \( f^{-1}(y) \) compact for each \( y \in Y \), therefore \( f^{-1}(K) \) is a compact subset of \( X \) contained in the open set \( f^{-1}(U) \). Let \( R \) be a finite union of members of such that \( f^{-1}(K) \subseteq R \subseteq f^{-1}(U) \). Thus \( K \subseteq f(R) \subseteq U \) and \( f(R) \) is a finite union of members of \( \mathcal{W} \). Hence \( Y \) is an \( \aleph \)-space, since it is regular also as \( X \) regular.

**DEFINITION 4.** [Doyle and Hocking, 2]. A space \( X \) is said to be an invertible space if for each open subset \( U \) of \( X \) there is a homeomorphism \( h : X \rightarrow X \) such that \( h(X - U) \subseteq U \). \( h \) is called an inverting homeomorphism for \( U \).

**THEOREM 23.** Let \( X \) be a topological space invertible in one of its non-empty open subsets \( U \) and let \( \overline{U} \) be an \( \aleph \)-space, then \( X \) is an \( \aleph \)-space.

**PROOF.** Let \( f \) be an inverting homeomorphism for \( U \). Then \( f(\overline{U}) \) is closed and \( X = \overline{U} \cup f(\overline{U}) \). Since \( \overline{U} \) and \( f(\overline{U}) \) are \( \aleph \)-spaces, therefore by Theorem 1, \( X \) is an \( \aleph \)-space.

**DEFINITION 5.** [Levine, 12]. Let \( (X, \tau) \) be any topological space. Then the topology \( \tau(A) = \{ U \cup (V \cap A) : U, V \in \tau \} \) where \( A \in \tau \), is called a simple extension for \( \tau \). Obviously \( A \in \tau(A) \). As is easily verified, \( (A, \tau \cap A) = (A, \tau(A) \cap A) \) and \( (X - A, \tau \cap (X - A)) = (X - A, \tau(A) \cap (X - A)) \).

**THEOREM 24.** Let \( (X, \tau) \) be an \( \aleph \)-space and \( A \) be a closed subspace of \( (X, \tau) \). Then \( (X, \tau(A)) \) is an \( \aleph \)-space.

**PROOF.** Since \( (X, \tau) \) is an \( \aleph \)-space, therefore \( (A, \tau \cap A) \) and \( (X - A, \tau \cap (X - A)) \) are \( \aleph \)-spaces. But \( (A, \tau \cap A) = (A, \tau(A) \cap A) \) and \( (X - A, \tau(A) \cap (X - A)) = ((X - A), \tau \cap (X - A)) \). Thus \( X \) is the union of two \( \tau(A) \)-closed \( \aleph \)-spaces \( A \) and \( X - A \). Hence \( (X, \tau(A)) \) is an \( \aleph \)-space.
M.K. Singal and Pushpa Jain

Institute of Advanced Studies, Meerut University, and Meerut (India).

Maitreyi College, Netaji Nagar, New Delhi-23, (India)

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