INTEGRABILITY CONDITIONS OF AN ALMOST CONTACT MANIFOLD

By R.S. Mishra

1. Introduction.

Let \( V^n \) be an \( n \)-dimensional differentiable manifold. Let there be defined in \( V^n \), a \( C^\infty \) vector valued linear function \( F \), a vector field \( T \) and a 1-form \( A \) satisfying

\[
(X + X) = A(X)T,
\]

for an arbitrary vector field \( X \), where

\[
X = F(X).
\]

Then \( V^n \) is called an almost contact manifold. It can be easily proved that

\[
\begin{align*}
(1.2) & \quad n \text{ is odd dimensional } = 2m + 1 \\
(1.3) & \quad \text{rank}(F) = 2m, \\
(1.4) & \quad T = 0, \\
(1.5) & \quad A(T) = 1, \\
(1.6) & \quad A(X) = 0,
\end{align*}
\]

for an arbitrary vector field \( X \).

Agreement (1.1). In the proceeding and in what follows the equations containing \( X, Y, Z, U \) etc. hold for arbitrary vector fields \( X, Y, Z, U \).

Let there be defined in \( V^n \) a metric tensor \( g \) satisfying

\[
g(X, Y) = g(X, Y) - A(X) A(Y).
\]

Then the almost contact manifold \( V^n \) is called an almost Grayan manifold.

Let us put in the almost Grayan manifold \( V^n \)

\[
(1.8) \quad \tilde{F}(X, Y) = g(X, Y).
\]

Then \( \tilde{F} \) is skew symmetric:

\[
\begin{align*}
(1.9) a & \quad \tilde{F}(X, Y) + \tilde{F}(Y, X) = 0 \\
\end{align*}
\]

and

\[
\begin{align*}
(1.9) b & \quad \tilde{F}(X, Y) = \tilde{F}(X, Y).
\end{align*}
\]

If in the almost Grayan manifold \( V^n \)

\[
(1.10) a \quad \tilde{F} = dA.
\]
Then $V_n$ is called an almost Sasakian manifold. Thus for an almost Sasakian manifold
\[(1.10)\]
\[
F(X, Y) = (dA)(X, Y)
\]
equivalent to
\[(1.10)\]
\[
F(X, Y) = (D_X A)(Y) - (D_Y A)(X),
\]
where $D$ is a symmetric connexion.

It is easy to see that for an almost Sasakian manifold
\[(1.11)\]
\[
(D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = 0.
\]
Nijenhuis tensor $N$ is given by
\[
N(X, Y) = [X, Y] + [\overline{X}, Y] - [X, \overline{Y}] - [\overline{X}, \overline{Y}].
\]

An almost contact manifold for which
\[(1.12)\]
\[
N(X, Y) + (dA)(X, Y) T = 0.
\]
holds is called an almost contact normal manifold.

Let us put
\[(1.13)\]
\[
I(X) = X - A(X) T,
\]
\[(1.14)\]
\[
m(X) = A(X) T.
\]
Then
\[(1.15)\]
\[
X = I(X) + m(X).
\]
It can be proved easily that
\[(1.16)\]
\[
I(\overline{X}) = I(\overline{X}) = \overline{X},
\]
\[(1.16)\]
\[
I(X) = I(\overline{X}) = - I(X),
\]
\[(1.17)\]
\[
m(\overline{X}) = m(\overline{X}) = 0,
\]
\[(1.18)\]
\[
l(m(X)) = m(l(X)) = 0,
\]
\[(1.19)\]
\[
I^2(X) = I(l(X)) = l(X),
\]
\[(1.20)\]
\[
m^2(X) = m(X)
\]
\[(1.21)\]
\[
l(T) = 0, m(T) = T.
\]
The operators $l$ and $m$ applied to the tangent space at each point of the manifold are complementary projection operators. Thus there exist in the manifold two complementary distributions $\Pi_{2m}$ and $\Pi_1$ corresponding to $l$ and $m$ respectively. $\Pi_{2m}$ is $2m$-dimensional and $\Pi_1$ is 1-dimensional.

2. Integrability conditions.

THEOREM (2.1). The distribution $\Pi_1$ is integrable.

PROOF. The distribution $\Pi_1$ is given by
\[(2.1)\]
\[
a) X = m(X),
\]
\[
b) I(X) = 0.
\]
In order that $\Pi_1$ is integrable, it is necessary and sufficient that

\[(d\Pi)(X, Y) = 0\]  

be satisfied by (2.1)a. Thus we have

\[(d\Pi)(m(X), m(Y)) = 0.\]

In consequence of (1.18), this equation is equivalent to

\[l([m(X), m(Y)]) = 0.\]

In consequence of (1.13) and (1.14) this equation is automatically satisfied. Hence we have the statement.

**THEOREM (2.2).** The necessary and sufficient condition that $\Pi_{2m}$ be integrable is

\[(d\Pi)(X, Y) = 0\]

equivalent to

\[(d\Pi)(X, Y) = 0.\]

**PROOF.** The distribution $\Pi_{2m}$ is given by

\[[a) X = i(X), \quad b) m(X) = 0,\]

In order that $\Pi_{2m}$ is integrable it is necessary and sufficient that

\[(d\Pi)(X, Y) = 0\]

be satisfied by (2.6)a. Hence, we have

\[(d\Pi)(i(X), i(Y)) = 0.\]

In consequence of (1.18), this equation is equivalent to

\[m([i(X), i(Y)]) = 0.\]

With the help of (1.13) and (1.14) this equation takes the form

\[A(X)[T(A(Y)) - A([T, Y])] - A(Y)[T(A(X)) - A([T, X])] = X(A(Y)) - Y(A(X)) - A([X, Y]),\]

which is the equation (2.5)a.

Barring $X$ and $Y$ in (2.5)a and using (1.6), we get (2.5)b.

**COROLLARY (2.1).** The equation (2.5)b is also equivalent to

\[(d\Pi)(X, Y) = 0\]

or

\[(d\Pi)(X, Y) = 0.\]

**PROOF.** In consequence of (1.6), the equation (2.5)b is equivalent to

\[(d\Pi)(X, Y) = 0,\]

which, by virtue of the definition of $N$ is the same as (2.5)c. Barring $X$ and $Y$
in (2.5)c, we get (2.5)d. Barring \( X \) and \( Y \) in (2.5)d, we get (2.5)c. We similarly obtain (2.5)e. Barring \( X \) in (2.5)e we get (2.5)c. Hence (2.5)c, d, e are equivalent.

**THEOREM (2.3).** Necessary and sufficient condition that \( V_n \) be integrable is (2.5).

**PROOF.** The statement follows from Theos. (2.1) and (2.2) and Cor. (2.1).

**COROLLARY (2.2).** If an almost contact manifold \( V_n \) is integrable


**PROOF.** The equation follows from (2.5)a, by using the fact that \( d^2 = 0 \).

**THEOREM (2.4).** The necessary and sufficient condition that an almost contact normal manifold be integrable is

\[(2.7) \ a \quad N(X, Y) = A(X)N(T, Y) - A(Y)N(T, X),\]

equivalent to

\[(2.7) \ b \quad N(X, \bar{Y}) = 0,\]

or

\[(2.7) \ c \quad N(X, \bar{Y}) = A(X)N(T, \bar{Y}).\]

**PROOF.** Substituting from (1.12) in (2.5)a, b we obtain (2.7)a, b. Barring \( Y \) in (2.7)a and using (1.6), we obtain (2.7)c.

**THEOREM (2.5).** An almost Sasakian manifold cannot be integrable.

**PROOF.** Substituting from (1.10)b in (2.5)a, and using \( 'F(T, Y) = 0 \), we get \( 'F = 0 \),

equivalent to

\[F = 0,\]

which proves the statement.