A PROPERTY OF COFUNCTORS $S_F(X, A)$

By Kwang Ho So

Abstract.

A $k$-dimensional vector bundle is a bundle $\xi=(E, P, B, F^k)$ with fibre $F^k$ satisfying the local triviality, where $F$ is the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ ([1], [2] and [3]). Let $\text{Vect}_k(X)$ be the set consisting of all isomorphism classes of $k$-dimensional vector bundles over the topological space $X$. Then $\text{Vect}_k(X) = \{\text{Vect}_k(X)\}_{k=0,1,\ldots}$ is a semigroup with Whitney sum (§1).

For a pair $(X, A)$ of topological spaces, a difference isomorphism over $(X, A)$ is a vector bundle morphism ([2], [3]) $\alpha: \xi_0 \rightarrow \xi_1$ such that the restriction $\alpha: \xi_0|A \rightarrow \xi_1|A$ is an isomorphism. Let $S_F(X, A)$ be the set of all difference isomorphism classes over $(X, A)$ of $k$-dimensional vector bundles over $X$ with fibre $F^k$. Then $S_F(X, A) = \{S_F(X, A)\}_{k=0,1,\ldots}$ is a semigroup with Whitney sum (§2).

In this paper, we shall prove a relation between $\text{Vect}_k(X)$ and $S_F(X, A)$ under some conditions (Theorem 2, which is the main theorem of this paper). We shall use the following theorem in the paper.

**Theorem 1.** Let $\xi=(E, P, B)$ be a locally trivial bundle with fibre $F$, where $(B, A)$ is a relative CW-complex. Then all cross sections $S$ of $\xi|A$ prolong to a cross section $S^\ast$ of $\xi$ under either of the following hypothesis:

(H1) The space $F$ is $(m-1)$-connected for each $m \leq \dim B$.

(H2) There is a relative CW-complex $(Y, X)$ such that $B=Y \times I$ and $A=(X \times I) \cap (Y \times 0)$, where $I=[0,1]$. (For proof see p.21 [2]).

1. Cofunctors $\text{Vect}_k$. 

Let $\xi=(E(\xi), P_\xi, B)$ and $\eta=(E(\eta), P_\eta, B)$ be two vector bundles over $B$. We define $\xi \oplus \eta$ by

$$E(\xi \oplus \eta) = \bigcup_{b \in B} E_b \oplus \eta_b$$

then, with a suitable topology on $E(\xi \oplus \eta)$, $\xi \oplus \eta = (E(\xi \oplus \eta), P_\xi \oplus P_\eta, B)$
is a vector bundle over $B$, where $\xi_b = P^{-1}(b)$ and $\eta_b = P^{-1}(b)$. (1), [2]). We call $\xi \oplus \eta$ the Whitney sum of $\xi$ and $\eta$.

For a continuous map $f: A \rightarrow B$, and a vector bundle $\xi$ over $B$, we define the fibre products bundle $f^*(\xi)$ induced by $f$ as follows,

$$ E(f^*(\xi)) = \{(x, y) \in A \times E(\xi) | f(x) = P(y)\}, $$

$$ P_{f^*(\xi)}: E(f^*(\xi)) \rightarrow A $ is defined by $P_{f^*(\xi)}((x, y)) = x$. 

Then, $f^*(\xi) = (E(f^*(\xi)), P_{f^*(\xi)}, A)$ is a vector bundle over $A$ ([1], [3]).

Let $\text{Pa}$ be the category of paracompact spaces and homotopy classes of maps, and let $\text{Ens}$ be the category of sets and functions. We take $[f]: A \rightarrow B$ in $\text{Pa}$. For $f, g \in [f]$ and a vector bundle $\xi$ over $B$, we can prove that $f^*(\xi) \cong g^*(\xi)$ (2).

Now, we define a function $\text{Vect}_k: \text{Pa} \rightarrow \text{Ens}$ by $\text{Vect}_k([f]) : \text{Vect}_k(B) \rightarrow \text{Vect}_k(A)$ such that $[\text{Vect}_k([f])](\xi) = (f^*(\xi))$ in $\text{Ens}$ for $[f]: A \rightarrow B$ in $\text{Pa}$, where $\xi \in \text{Vect}_k(B)$ and $(f^*(\xi)) \in \text{Vect}_k(A)$ are the isomorphism classes containing $\xi$. For $f, g \in [f]: A \rightarrow B$, since $f^*(\xi) \cong g^*(\xi)$ $\text{Vect}_k([f])$ is well defined.

**PROPOSITION 1.** The family of functions $\text{Vect}_k: \text{Pa} \rightarrow \text{Ens}$ is a cofunctor.

**PROOF.** For $1_B: B \rightarrow B$, $B \in \text{Pa}$, and a vector bundle $\xi$ over $B$, $1_B^*(\xi) \cong \xi$ and therefore $\text{Vect}_k(1_B)$ is the identity. If $g \circ f: B_2 \rightarrow B_1 \rightarrow B$ is continuous, where $B_2, B_1$ and $B \in \text{Pa}$, for a vector bundle $\xi$ over $B$, $(g \circ f)^*(\xi) \cong g^*(f^*(\xi))$ ([2]). Thus $\text{Vect}_k([g \circ f]) = \text{Vect}_k([f] \circ \text{Vect}_k([g])$. Therefore $\text{Vect}_k$ is a cofunctor.

*The Stiefel variety of orthogonal k-frames in $\mathbb{R}^n$, written $V_k(\mathbb{R}^n)$, is the subspace of $(v_1, \ldots, v_k) \in (S^{n-1})^k$ such that $v_i \perp v_j$ for $i \neq j$. The Grassmann variety of k-dimensional subspaces of $\mathbb{R}^n$, written $G_k(\mathbb{R}^n)$, is the set of k-dimensional subspaces of $\mathbb{R}^n$ with the quotient topology defined by the function $(v_1, \ldots, v_k) \rightarrow \langle v_1, \ldots, v_k \rangle$ of $V_k(\mathbb{R}^n)$ onto $G_k(\mathbb{R}^n)$, where $\langle v_1, \ldots, v_k \rangle$ is the k-dimensional subspace of $\mathbb{R}^n$ with basis $v_1, \ldots, v_k$ ([2]). We have the canonical k-dimensional vector bundle $\gamma_k^n$ over $G_k(\mathbb{R}^n)$, which is defined by $E(\gamma_k^n) = \{(V, x) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n | x \in V\}$, $P_{\gamma_k^n}: E(\gamma_k^n) \rightarrow G_k(\mathbb{R}^n)$ is the projection on the first argument, that is $\gamma_k^n = (E(\gamma_k^n), P_{\gamma_k^n}, G_k(\mathbb{R}^n))$. $\gamma_k^n$ is the vector bundle with fibre $\mathbb{R}^k$, i.e $\gamma_k^n$ is the
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$k$-dimensional vector bundle over $G_k(R^n)$.

Suppose a space $B$ is in $\mathbf{P}$. Every $k$-dimensional vector bundle $\xi_k$ over $B$ is isomorphic to $f^*(\gamma_k^{\infty})$ for some continuous map $f: B \to G_k(F^{\infty})$ as vector bundles over $F$ (p. 31 of [2]). If we put $[B, G_k(F^{\infty})]$ the set of homotopy classes of maps from $B$ to $G_k(F^{\infty})$, then there exists an one-to-one correspondence between $[B, G_k(F^{\infty})]$ and $\text{Vect}_k(B)$. This function $\phi_B: [B, G_k(F^{\infty})] \to \text{Vect}_k(B)$ is defined by $\phi_B([f]) = \{f^*(\gamma_k^{\infty})\}$ for $[f] \in [B, G_k(F^{\infty})]$. Since $[\quad, G_k(F^{\infty})]: \mathbf{P} \to \text{Ens}$ is a cofunctor, in fact $\phi$ is a natural transformation between cofunctors $[\quad, G_k(F^{\infty})]$ and $\text{Vect}_k$, where $\phi$ is the family of functions $\phi_B$ for $B \in \mathbf{P}$.

**PROPOSITION 2.** $\phi$ is a natural equivalence from $[\quad, G_k(F^{\infty})]$ to $\text{Vect}_k$.

**PROOF.** At first, to prove that $\phi$ is a natural transformation, we take a homotopy class $[f]: B_1 \to B$ of maps in $\mathbf{P}$. Then we have the commutative diagram:

$$
\begin{array}{ccc}
[B, G_k(F^{\infty})] & \phi_B & \text{Vect}_k(B) \\
[f|, G_k(F^{\infty})] & \circ & \text{Vect}_k([f]) \\
[B_1, G_k(F^{\infty})] & \phi_{B_1} & \text{Vect}_k(B_1)
\end{array}
$$

that is, for $[g] \in [B, G_k(F^{\infty})]$

$$
\begin{align*}
\text{Vect}_k([f]) \circ (\phi_B([g])) &= \text{Vect}_k([f]) \circ (\gamma_k^{\infty}) = \{f^* \circ g^*(\gamma_k^{\infty})\} \\
\phi_{B_1}([f], G_k(F^{\infty}))(g) &= \phi_{B_1}([g], f) = \phi_{B_1}([g] \circ f) = \{g \circ f\} \circ (\gamma_k^{\infty}) = \{f^* \circ g^*(\gamma_k^{\infty})\}.
\end{align*}
$$

For each $B \in \mathbf{P}$, $\phi_B$ is surjective and injective ([2]), and therefore $\phi$ is a natural equivalence.

2. Cofunctor $S_F(X, A)$.

Suppose a pair $(X, A)$ of spaces. Two difference isomorphisms over $(X, A)$, $\alpha: \xi_0 \to \xi_1$ and $\beta: \eta_0 \to \eta_1$ are isomorphic if there exist isomorphisms $u_i: \xi_i \to \eta_i$ (over $X$) for $i = 0, 1$ such that the following diagram of isomorphisms is commutative.

$$
\begin{array}{ccc}
\xi_0|A & \alpha & \xi_1|A \\
\eta_0|A & \circ & \eta_1|A
\end{array}
$$
Let $S_k(X,A)$ be the set of all difference isomorphism classes of $k$-dimensional, $F$-vector bundles over $(X,A)$. For a continuous function $f: (X,A) \to (Y,B)$ with $f(A) \subset B$, we define $S_k(f): S_k(Y,B) \to S_k(X,A)$ by $[S_k(f)](\xi) = (f^*(\xi))$, where $\{\xi\} \in S_k(Y,B)$ and $(f^*(\xi)) \in S_k(X,A)$.

**Proposition 3.** If $f: (X,A) \to (Y,B)$ is a continuous map such that $f(A) \subset B$, and if $\beta: \eta_0 \to \eta_1$ is a difference isomorphism over $(Y,B)$, then $f^*(\beta): f^*(\eta_0) \to f^*(\eta_1)$ is a difference isomorphism over $(X,A)$.

**Proof.** We want to show that $f^*(\beta): f^*(\eta_0)|A \to f^*(\eta_1)|A$ is an isomorphism. For the inclusion map $i: A \to X$, $(f \circ i)^*\eta_0 = i^*f^*\eta_0 \equiv i^*f^*(\eta_0)|B \equiv f^*(\eta_0)|A$ and $(f \circ i)^*\eta_1 = i^*f^*\eta_1 \equiv i^*f^*(\eta_1)|B \equiv f^*(\eta_1)|A$, and therefore $f^*(\beta): f^*(\eta_0)|A \to f^*(\eta_1)|A$ is an isomorphism, because of $(f \circ i)^*(\beta): (f \circ i)^*(\eta_0)|B \to (f \circ i)^*(\eta_1)|B$ is an isomorphism induced from the isomorphism $\beta: \eta_0|B \to \eta_1|B$.

By this proposition we see that $S_k(f)$ is well defined. Let $C \times C_0$ be the category of all pairs of topological spaces and maps between pairs. Then

$$S_k: C \times C_0 \to \text{Ens}$$

is a cofunctor. Put $S_k(X,A) = \{S_k(X,A)\}_{k=0,1,\ldots}$. Then

$$S_k: C \times C_0 \to \text{Ens}$$

is a cofunctor. We define a commutative semigroup structure on $S_k(X,A)$, using the quotient function of the Whitney sum operation defined as usual by $\alpha \oplus \beta: \xi_0 \oplus \eta_0 \to \xi_1 \oplus \eta_1$ for $\alpha: \xi_0 \to \xi_1$ and $\beta: \eta_0 \to \eta_1$. Of course, if $\alpha: \xi_0 \to \xi_1$ and $\beta: \eta_0 \to \eta_1$ are difference isomorphisms over $(X,A)$, then $\alpha \oplus \beta: \xi_0 \oplus \eta_0 \to \xi_1 \oplus \eta_1$ is a difference isomorphism over $(X,A)$. Let $S_g$ be the category of all semigroups and semigroup maps. Then

$$S_g: C \times C_0 \to S_g$$

is a cofunctor.

### 3. The main theorem.

In this section, we assume that $A$ be a subcomplex or subspace of a finite CW-complex $X$.

**Theorem 2.** (Main theorem) For $(X,A)$, $S_k(X,A)$ is a sub-semigroup of $\text{Vect}_F(A)$. If $X$ is deformable into a subspace $A$, then $S_k(X,A) \cong \text{Vect}_F(A)$ as
semigroups.

To prove this theorem we need the following lemmas.

**Lemma 1.** Let \( \xi_0 \) and \( \xi_1 \) be two vector bundles over \( X \). If \( u: \xi_0|A \rightarrow \xi_1|A \) is a vector bundle morphism, then there exists a unique vector bundle morphism \( v: \xi_0 \rightarrow \xi_1 \) such that \( v|A = u \), where \( A \) is a subcomplex.

**Proof.** Define \( \text{Hom}_F(\xi_0, \xi_1)_x = \bigcup_{x \in X} \text{Hom}_F((\xi_0)_x, (\xi_1)_x) \), where \( (\xi_0)_x \) is the fibre at \( x \in X \) of \( \xi_0 \), and so on. Since each \( \text{Hom}_F((\xi_0)_x, (\xi_1)_x) \) for all \( x \in X \) is a vector space, \( \text{Hom}_F(\xi_0, \xi_1) = (\text{Hom}_F(\xi_0, \xi_1)_x) \), \( P, X \) is a vector bundle over \( X \), where \( P^{-1}(x) = \text{Hom}_F((\xi_0)_x, (\xi_1)_x) \). Then we can view \( u \) as a cross section of \( \text{Hom}_F(\xi_0, \xi_1) \) over \( A \), i.e., for each \( x \in A \), \( u(x) = u|_{(\xi_0)_x}: (\xi_0)_x \rightarrow (\xi_1)_x \). Since every vector space is contractible, the fibre of \( \text{Hom}_F(\xi_0, \xi_1) \) is also contractible. Therefore, by Theorem 1, \( u \) is extended to a unique cross section \( v: X \rightarrow \text{Hom}_F(\xi_0, \xi_1) \). In this case, \( v = \{v(x): (\xi_0)_x \rightarrow (\xi_1)_x \mid x \in X \} \) is a vector bundle morphism \( v: \xi_0 \rightarrow \xi_1 \) which prolongs \( u \).

**Lemma 2.** If \( X \) is deformable into \( A \), then \( \text{Vect}_F(X) \cong \text{Vect}_F(A) \) as semigroups.

**Proof.** Since \( X \) is deformable into \( A \), there is a continuous map \( f: X \rightarrow A \) such that \( i\circ f \sim 1_X \) (homotopic), where \( i: A \rightarrow X \) is the inclusion map \( (4) \). Recall that there is an one-to-one correspondence between \([B, G_k(F^\infty)]\) and \( \text{Vect}_F(B) \) for \( B \in \text{Pa} \) and for all \( k = 0, 1, \ldots \).

In the diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{g} & G_k(F^\infty) \\
\downarrow{\text{deformation = } f} & & \downarrow{f} \\
A & \xrightarrow{h} & G_k(F^\infty)
\end{array}
\]

We see that \( g \circ f = f|A \circ g\) and \( h \circ f = h|A \circ f \), because of \( f \sim 1_X \) implies that \( f|A \approx 1_A \).

Thus there is the one-to-one correspondence

\[
\phi: \ [X, G_k(F^\infty)] \rightarrow [A, G_k(F^\infty)]
\]

defined by \( \phi([g]) = [g|A] \). The inverse \( \phi^{-1} \) of \( \phi \) is defined by \( \phi^{-1}([h]) = [h|f] \). Then \( \phi^{-1} \circ \phi([g]) = \phi^{-1}([g|A]) = [g|A \circ f] = [g] \) and \( \phi \circ \phi^{-1}([h]) = \phi([h|f]) = [h|A] = [h] \).

Therefore we define \( \overline{\phi}: \text{Vect}_F(X) \rightarrow \text{Vect}_F(A) \) by \( \overline{\phi}([g^*\gamma_k^\infty]) = ([g|A] \cdot \gamma_k^\infty) \).

Then \( \overline{\phi} \) is a semigroup isomorphism.
Proof of Theorem 2. Let \( \xi_0 \) and \( \xi_1 \) be two vector bundles over \( X \). By Lemma 1, if there is a vector bundle isomorphism \( \xi_0 \mid A \rightarrow \xi_1 \mid A \) over \( A \) then \( \{ \xi_0 \mid A \} = \{ \xi_1 \mid A \} \) in \( S_F(X, A) \). Therefore the morphism

\[
\phi_F : S_F(X, A) \rightarrow \text{Vect}_F(A)
\]

defined by \( \phi_F(\{ \xi \}) = \{ \xi \mid A \} \) is injective and preserves Whitney sum. (Note that the class \( \{ \xi \} \) in \( \phi_F(\{ \xi \}) \) is the difference isomorphism class containing \( \xi \) and the \( \{ \xi \mid A \} \) is the isomorphism class containing \( \xi \mid A \). Thus \( \phi_F \) is a monomorphism between semigroups, and therefore \( S_F(X, A) \) is isomorphic to a sub-semigroup of \( \text{Vect}_F(A) \).

Let us assume that \( X \) is deformable into \( A \). Then there is a continuous map \( f : X \rightarrow A \subset X \) such that \( f = 1_X \). Lemma 2 says that \( \xi_0 \cong \xi_1 \) over \( X \) iff \( \xi_0 \mid A \cong \xi_1 \mid A \) over \( A \) in our situation. That is, there is a semigroup isomorphism \( \phi_X : \text{Vect}_F(X) \cong S_F(X, A) \) defined by \( \phi_X(\{ \xi \}) = \{ \xi \} \). Define \( \phi_F^{-1} : \text{Vect}_F(A) \rightarrow S_F(X, A) \) by the commutative diagram:

\[
\begin{array}{ccc}
\text{Vect}_F(A) & \xrightarrow{\phi_F} & S_F(X, A) \\
\phi_F^{-1} \downarrow & & \downarrow \phi_X \\
\text{Vect}_F(X) & & \\
\end{array}
\]

Then \( \phi_F^{-1} \circ \phi_F = 1_{S_F(X, A)} \) and \( \phi_F \circ \phi_F^{-1} = 1_{\text{Vect}_F(A)} \).

Let us denote the completions of \( \text{Vect}_F(A) \) and \( S_F(X, A) \) by \( K_F(A) \) and \( KS_F(X, A) \), respectively \((2)\). Then, from Theorem 2 we easily obtain the following.

Corollary 1. \( KS_F(X, A) \) is a subgroup of \( K_F(A) \). If \( X \) is deformable into \( A \), then \( KS_F(X, A) \cong K_F(A) \) as abelian groups.

Jeonbuk University
Korea

References