A SUMMATION FORMULA ON KAMPÉ DE FÉRIET FUNCTION

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During the course of investigation, an attempt has been made by several workers to obtain the exact solution of many problems in quantum mechanics in terms of Appell's functions; in an extension of this work, the author establishes here a summation formula for Kampé de Fériet's function, which may prove to be useful. The formula is as follows:

\[
\sum_{m=0}^{n} \frac{\Gamma(k+m+\frac{1}{2}) (2yz)^{m}}{m!} \binom{-k-m}{2} \frac{\Delta(2,-m); \Delta(2,-m)}{\Delta(2,-2k-2m)} \frac{y^{-2}}{z^{-2}} \]

where "\(\Rightarrow\)" shows the presence of a similar term with \(y\) and \(z\) interchanged.

1. Introduction.

Kampé de Fériet J. [\((1), p.150\)] has introduced a generalized hypergeometric function of two variables in the form

\[
\binom{a}{b} \binom{x}{y} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{m}{r!s!} \left( \prod_{j=1}^{n} (a_j)_{r+s} \right) \left( \prod_{j=1}^{p} (b_j)_{r+s} \right) x^r y^s,
\]

where \(a_m \) stands for \(a_1, a_2, \ldots, a_m\); \( \prod_{j=1}^{m} (a_j)_{r} \) represents the product \( (a_1)_r (a_2)_r \cdots (a_m)_r \) and for absolute convergence of the series \(|x| < 1, |y| < 1, m+l \leq n+p+1\).

The aim of this note is to establish a summation formula for Kampé de Fériet's function which may be utilized in the exact solution of a large number of
problems of Mathematics, both pure and applied, and in mathematical physics frequently expressed in terms of Kampé de Fériet's functions. In a recent paper [(3), p.9, (4.5)] the author has obtained the formal solution of a problem of heat conduction equation in terms of Kampé de Fériet's functions.

2. A Summation Formula.

We start from the known result [(3), p.5, (2.1) for \( n=0 \):]

\[
\left(2.1\right) \int_{-\infty}^{\infty} e^{-x^2} x^{2k} \left[ \frac{1}{x_{p+2} F_q \left( \Delta(2, -l), A; B, \mu x^2 \right)} \right] dx
\]

\[
= \Gamma \left( k+\frac{1}{2} m+\frac{1}{2} l, \frac{1}{2} \right) \left[ \begin{array}{c} p+2 \cr 2 \cr q \end{array} \right] \Delta(2, -l), A; \Delta(2, -m), a_p \]

\[
\left(2.2\right) \int_{-\infty}^{\infty} e^{-x^2} x^{2k} H_m(yx) H_m(zx) dx = \Gamma \left( k+m+\frac{1}{2} \right) (4yz)^m
\]

where \( l, m \) are positive integers and \( \Delta(m, n) = \frac{n+1}{m}, \ldots, \frac{n+m-1}{m} \).

Further setting \( p=q=0, \mu=-y^2, \lambda=-z^2, l=m \) etc, in (2.1), we get

\[
\left(2.2\right) \int_{-\infty}^{\infty} e^{-x^2} x^{2k} H_m(yx) H_m(zx) dx = \Gamma \left( k+m+\frac{1}{2} \right) (4yz)^m
\]

where \( H_n(x) = (2x)^n \left( \begin{array}{c} \Delta(2, -n), -x^2 \end{array} \right) \) is the Hermite polynomial. Therefore

\[
\sum_{m=0}^{n} \frac{\Gamma \left( k+m+\frac{1}{2} \right) (2yz)^m}{m!} \left[ \begin{array}{c} 1 \cr 2 \cr 0 \end{array} \right] \Delta(2, -m); \Delta(2, -m) \]

\[
\int_{-\infty}^{\infty} e^{-x^2} x^{2k} H_m(yx) H_m(zx) dx.
\]

The change in order of summation and integration is easily justified and we have the R.H.S. as

\[
= \int_{-\infty}^{\infty} e^{-x^2} x^{2k} \left( \sum_{m=0}^{n} \frac{H_m(yx) H_m(zx)}{2^m m!} \right) dx.
\]
A Summation Formula On Kampé de Fériet Function

Using, Christoffel-Darboux formula \([2], p. 193, (11)\):

\[
\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} = \frac{H_{n+1}(x)H_n(y) - H_{n}(x)H_{n+1}(y)}{2^{n+1} n! (x-y)}.
\]

the R.H.S. becomes

\[
\frac{1}{2^{n+1} n!} \int_{-\infty}^{\infty} e^{-x^2 - y^2} \cdot x^{-2n} (H_{n+1}(zx)H_n(yz) - H_n(xz)H_{n+1}(yz)) dx.
\]

Now separating the R.H.S. as the difference of two integrals then in view of \((2.1)\), we obtain

\[
(2.3) \sum_{m=0}^{n} \frac{\Gamma(k+m+\frac{1}{2})(2yz)^m}{m!} F \left[ \begin{array}{c} 1 \\ 2 \\ 0 \end{array} \right] \frac{-k-m}{\Delta(2, -m) \Delta(2, -m)} \frac{y^{-2}}{z^{-2}}
\]

\[
= \frac{(2yz)^n \Gamma(k+n+\frac{1}{2})}{n!(y-z)} \left\{ yF \left[ \begin{array}{c} 1 \\ 2 \\ 0 \end{array} \right] \frac{-k-n}{\Delta(2, -n-1) \Delta(2, -n)} \frac{y^{-2}}{z^{-2}} \right\}
\]

where “\(\ldots\)” is used to indicate the presence of a similar term with \(y\) and \(z\) interchanged. This is the summation formula.

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REFERENCES