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On σ-System of L-Set

by

Ji-Ho Joung

Dong Guk University, Seoul, Korea

1. Introduction

The length l(I) of an interval I is defined, as usual, to be the difference of the end points of the interval. Length is an example of a set function; that is, a function which associates an extended real number to each set in some collection which associates an extended real number to each set in some collection of sets. In the case of length the domain is the collection of all intervals. We should like to extended the notion of length to more complicated sets than intervals. For instance, we could define the "length" of an open set to be the sum of the lengths of the open intervals of which it is composed. Since the class of open sets is still too restricted for our purposes, we would like to construct a set function m which assigns to each set E in some collection H of sets of real numbers a nonegative extended real number mE called the measure of E.

n-dimentional Euclidean space R^n is when following the length

$$\varphi(x, y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots\cdots+(x_n-y_n)^2}$$

is defined. The whole of interval is length of segment in R', area in R^2, and volume in than R^3 generally and it is denoted by writing $\prod_{i=1}^{n}(b_i-a_i)$.

When j exist such that $b_j=a_j$, the volume of interval $a_i \leq x \leq b_i (i=1, 2, \cdots\cdots, n)$ in R^n is 0 and when j exist such that $b_j=\infty$, it defines $\prod_{i=1}^{n}(b_i-a_i)=\infty$.

If sum-set of sequence of intervals ω_i of a finite or a countable is presented the mass of interval ω, It is evident that the sum, the product, and the subtract etc. is mass of interval.

2. Lebesgue Measure

Definition 2.1. In n-dimentional Euclidean space R^n, for all point set e covering the sequence of intervals $\{\omega_i\}$ (finite sequence or infinite sequence), the infimum of volume of the sequence of intervals (notwithstanding method of division of the sequence of intervals)

$$\overline{m}e = \inf \sum_{i=1}^{\infty} m\omega_i, \quad e \subset U_i \omega_i$$

is called the outer measure of e.

We contract that the outer measure of empty set is 0. In reference to the outer measure following properties are formed.

Properties: ① For all point set e, the outer measure is an unique set function, and $0 \leq \overline{m}e \leq \infty$

② if, $e_1 \subset e_2$, $\overline{m}e_1 \leq \overline{m}e_2$.

From definition of outermeasure properties, ①, ② is obvious.

③ if $e = \bigcup_{i=1}^{\infty} e_i$, $\overline{m}e \leq \sum_{i=1}^{\infty} \overline{m}e_i$.

Proof Take for all $\varepsilon > 0$, and let

$$\varepsilon = \sum_{i=1}^{\infty} \varepsilon_i, \quad \varepsilon_i > 0 \qquad (1)$$

From definition of the outer measure on infimum, if there exist the sequence of intervals ω_{ij} covering e_i, that is,

$$e_i \subset \bigcup_{j=1}^{\infty} \omega_{ij},$$

then for this sequence of intervals $\{\omega_{ij}\}$.

$$\sum_{j=1}^{\infty} m\omega_{ij} \geq \overline{m}e_i \geq \sum_{i=1}^{\infty} m\omega_{ij} - \varepsilon_i,$$

so that

$$\sum_{i,j} \omega_{ij} \geq \sum_{i=1}^{\infty} \overline{m}e_i \geq \sum_{ij} m\omega_{ij} - \sum_{i=1}^{\infty} \varepsilon_i.$$

Thus, from (1)

$$\sum_{i=1}^{\infty} \overline{m}e_i \geq \sum_{i,j} m\omega_{ij} - \varepsilon. \qquad (2)$$

Whereas $e \subset \bigcup_{ij} \omega_{ij}$, and

ince this right side is the sum of the sequence of intervals, from definition

$$\overline{m}e \leq \sum_{i,j} m\omega_{ij} \qquad (3)$$

Where (2), (3) $\overline{m}e \leq \sum_{i,j} m\omega_{ij} \leq \sum_{i=1}^{\infty} \overline{m}e_i + \varepsilon$.

From, ε is arbitrary, so that

$$\overline{m}e \leq \sum_{i=1}^{\infty} \overline{m}e_i$$

※ Even though $e = \sum e_i$ is the sum of single sets, by property (3), the outer measure is not completely additive.

④ The outer measure of mass of interval ω equal to volume.

That is $\overline{m}\omega = m\omega$.

Proof : From the definition of the outer measure

$$\overline{m}\omega \leq m\omega \qquad (1)$$

now, if $\omega < \bigcup_i \omega_i$,

then $\omega = \bigcup_i (\omega\omega_i)$.

Thus $m\omega \leq \sum m(\omega\omega_i) \leq \sum m\omega_i$

$$m\omega \leq \inf \sum m\omega_i = \overline{m}\omega$$

that is, $m\omega \leq \overline{m}\omega$ (2)

In (1), (2) $m\omega = m\omega$.

Definition 2.2.

Let e be bounded, and the complement of e for the interval ω such that $e \subset \omega$ is e', then

$$\omega = e + e'.$$

If the outer measure $\overline{m}$ forms the additive, by property ④, $\overline{m}e + \overline{m}e' = \overline{m}\omega = m\omega$.

But since it is not necessary formed, Lebegue suppose that $\underline{m}e = \overline{m}\omega - \overline{m}e'$

$$\underline{m}e = m\omega - \overline{m}e',$$

and he defines that $\underline{m}e$ is the inner measure of set e.

particularly, When $\overline{m}e = \underline{m}e$, he defines that e is the measurable set, and $me = \overline{m}e = \underline{m}e$ is the measure of e. In this case, it is evident that $\overline{m}w = \overline{m}e + \overline{m}e'$ is formed.

This is equivalent to $\overline{m}e + \overline{m}e' = m\omega$.

Now, e lift restriction that bounded and for all intervals ω,

$$m\omega = \overline{m}(\omega e) + \overline{m}(\omega e') \tag{1}$$

which defines measurable set of e.

Caratheodory generalize (1) by much abstraction, e is not bounded, has not interval ω and for general e and for all set u,

$$\text{if } \overline{m}u = \overline{m}ue + \overline{m}ue' \tag{2}$$

then e is measurable and $me = \overline{m}e$ define by the measure of e.

By (2) defined set e of measurable set say L-Set and the "me"asure me say L-measure (Lebesgue measure).

Let us derive (2) from (1).

Let $u \subset \bigcup_i \omega_i$, $\overline{m}u > \sum m\omega_i - \varepsilon$, $(\varepsilon > 0)$.

By properties (1), (2)

$$\bigcup e \subset \bigcup_i w_i e$$

$$\overline{m}ue \leq \overline{m}(\bigcup_i \omega_i e) \leq \sum_{i=1}^{\infty} \overline{m}\omega_i e$$

$$ue' \subset \bigcup_i \omega_i e'$$

$$\overline{m}ue' \leq \overline{m}(\bigcup_i \omega_i e \leq) \sum_{i=1}^{\infty} \overline{m}\omega_i e'$$

Hence

$$\overline{m}ue + \overline{m}ue' \leq \sum_{i=1}^{\infty} (\overline{m}\omega_i e + \overline{m}\omega_i e')$$

then, from (1) for the interval ω_i,

$$m\omega_i = \overline{m}\omega_i e + \overline{m}\omega_i e'.$$

Thus $\overline{m}ue + \overline{m}ue' \leq \sum_{i=1}^{\infty} m\omega_i < mu + \varepsilon$

Where since $\varepsilon > 0$ is arbitrary,

$$\overline{m}ue + \overline{m}ue' \leq \overline{m}u \tag{3}$$

Now, since $u = ue + ue'$, by property ③

$$\overline{m}u \leq \overline{m}ue + \overline{m}ue' \tag{4}$$

In (3), (4) $\bar{m}u=\bar{m}ue+\bar{m}ue'$.

$\bar{m}u=\bar{m}ue+\bar{m}ue'$

3. σ-system of L-Set

Theorem 1. Every set ω of R^n is L-Set.

Proof) Caratheodory-definition, i.e. When $\bar{m}u+\bar{m}ue+\bar{m}ue'$ form, the measure of e become $me=\bar{m}e$, and $e\in L$.

Let us prove the theorem by application of the above. Since ω is for every set of R^n,

$$u\omega=u,\ \omega'=0\ \ (\omega=R^n,\ R'^n=0)$$

Hence we show that $\bar{m}u=\bar{m}u\omega+\bar{m}u\omega'$ is formed.

now, in right side $\bar{m}u\omega=\bar{m}u$

$\bar{m}u\omega'=\bar{m}0=0$ (From definition, the outer measare of null set is 0.)

Thus $\bar{m}u=\bar{m}u\omega+\bar{m}u\omega'$ is formed.

Hence $\omega\in L$.

That is, ω is L-set.

Theorem 2. Complement e' of L-set e is L-set.

Proof) Since $e\in L$, for the complement e' of e in the meaning that e is L set, $\bar{m}u=\bar{m}ue+\bar{m}ue'$. Since this equation is formed for e', that $e'\in L$.

Theorem 3. For e_1, e_2 such that $e_1\in L$, $e_2\in L$, $e=e_1\cup e_2$, $e=e_1\cap e_2$ is L-set respectively, and its difference is L-set.

Proof) First, we prove that $e=e_1\cup e_2\in L$. To prove $\bar{m}u=\bar{m}ue+\bar{m}ue'$, if we replace u with ue_1', and in the meaning that e_2 is L-set.

$$\bar{m}ue_1'=\bar{m}ue_1'e_2+\bar{m}ue_1'e_2' \tag{1}$$

and a substitute for ue and in the meaning that e_1 is L-set,

$$\bar{m}ue=\bar{m}uee_1+\bar{m}uee_1' \tag{2}$$

now

$$\left.\begin{aligned} &ee_1=(e_1\cup e_2)e_1=e_1\\ &ee_1'=(e_1\cup e_2)e_1'=e_1e_1'e_2\cup e_1'e_2=e_1'e_2\\ &e_1'e_2'=(e_1\cup e_2)'e' \end{aligned}\right\} \tag{3}$$

$$(3)\rightarrow(2):\ \bar{m}ue=\bar{m}ue_1+\bar{m}ue_1'=e_2 \tag{4}$$

$$(3)\rightarrow(1):\ \bar{m}ue_1'=\bar{m}u_1'e_2+\bar{m}ue' \tag{5}$$

$$(4)\rightarrow(5):\ \bar{m}ue-\bar{m}ue_1'=\bar{m}ue_1-\bar{m}ue'$$

Thus $\bar{m}ue+\bar{m}ue'=\bar{m}ue_1+\bar{m}ue_1'=\bar{m}u$.

That is, $\bar{m}u=\bar{m}ue+\bar{m}ue'$ is formed, hence $e \in L$.

i.e. $e = e_1 \cup e_2 \in L$.

Next, we prove that $e = e_1 \cap e_2 \in L$. From morgan's law,

$$e = \bigcap_{i=1}^{2} e_i = (\bigcup_{i=1}^{2} e_i')'.$$

Now, since $e_i \in L\,(i=1,2)$, by (2) $e_i' \in L\,(i=1,2)$. Thus by (3) $\bigcup_{i=1}^{2} e_i' \in L$.

Again, by (2) $(\bigcup_{i=1}^{2} e_i')' \in L$

that is, $\bigcap_{i=1}^{2} (e_i')' = \bigcap_{i=1}^{2} e_i = e_1 \cap e_2 \in L$

It is evident by before.

Because the difference set can present as the product set in set-operation.

Theorem 4. $\left.\begin{array}{l} e_i \in L\,(i=1,2,\cdots\cdots) \\ e = \sum_{i=1}^{\infty} e_i \end{array}\right\} \longrightarrow e \in L$ (completely additive)

Proof) If $e_1 \in L$, then $\qquad \overline{m}u = \overline{m}ue_1 + \overline{m}ue_1' \qquad (1)$

the same way, if $S_n = \sum_{i=1}^{n} e_i$

then $\qquad \overline{m}u = \sum_{i=1}^{n} \overline{m}ue_i + \overline{m}us_n' \qquad (2)$

By mathematical induction, We prove that this equation is formed.

In (2), though u substitute for us'_n., by the definition of the measure of Caratheodory for $e_{n+1} \in L$, $\qquad \overline{m}us_n' = \overline{m}us_n' e_{n+1} + \overline{m}us_n' e'_{n+1} \qquad (3)$

But $\begin{cases} s_n' e_{n+1} = e_{n+1} \\ s_n' e_{n+1} = s_{n+1} \end{cases}$

Thus (3) is $\overline{m}us_n' = \overline{m}ue_{n+1} + \overline{m}us'_{n+1}$. If this substitute for right side of (2), then

$$\overline{mu} = \sum_{i=1}^{n} \overline{mue_i} + \overline{mue_{n+1}} + \overline{mus'}_{n+1}$$

$$= \sum_{i=1}^{n+1} \overline{mue_i} + \overline{mus'}_{n+1}$$

Hence (2) is proved by mathematical induction.

New in (2), since $s_n \subset e$, $s_n' \supset e'$.

Thus $\overline{m}us_n' \geq \overline{m}ue'$,

(2) is $\overline{m}u \geq \sum_{i=1}^{n} \overline{m}ue_i + \overline{m}ue'$. $\qquad (4)$

Now since n is arbitrary, in (4)

$$\overline{m}u \geq \sum_{i=1}^{n} \overline{m}ue_i + \overline{m}ue'. \qquad (5)$$

Next, since $\qquad e = \sum_{i=1}^{\infty} e_i$, by property③, for $ue = \sum_{i=1}^{\infty} ue_i$,

$$\overline{m}ue \leq \sum_{i=1}^{\infty} \overline{m}ue_i \qquad (6)$$

In (5), (6) $\overline{m}u \geqq \overline{m}ue + \overline{m}ue'$ $\qquad (7)$

But $u = ue + ue'$

Thus $\bar{m}u \leq \bar{m}ue + \bar{m}ue'$ (8)

From (7), (8) $\bar{m}u = \bar{m}ue + \bar{m}ue'$

Hence $e \in L$, that is, it is fulfil the Caratheodory-Theorem.

Thoreom 5. The production and the summation of all sequence of sets of L-set is L-set.

Proof) $e = \bigcup_i e_i, \ e_i \in L$

$$e = e_1 + (e_2 - e_1) + (e_3 - e_1 - e_2) + \cdots\cdots + (e_i - e_1 - e_2 - \cdots\cdots - e_{i-1}) + \cdots\cdots$$

the right side is sum of simple sets and since by theorem 3, each item is L-set, by theorem 4, $e \in L$.

※ From theorem 1, 2, 5, L-set form σ-system

—— *References* ——

〔1〕 Dr.E.Kamke : Mengenlehre

〔2〕 Paul R. Halmos : Naive set Theory.

〔3〕 Goffmann : Instroduction to Real Analysis

〔4〕 H.L. Royden : Real Analysis

〔5〕 Halmos : Measure Theory

〔6〕 Kakutani : Measuse Theory

〔7〕 Munroe : Measure and Integration

〔8〕 Konto : Real Function

〔9〕 Takaki : Survey of Calculus

〔10〕 McShane. Botts : Real Analysis

〔11〕 Yoshita : Introduction to Lebesgue Integral

〔12〕 Komass : Lebesgue Integral.

〔13〕 Koriki : Theory of Real Function and Theory of Integration

〔14〕 Sji : Theory of Real Function