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THE PRIME RADICAL OF A POWER SERIES RING

by

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It is known that Rad $R[x]=(\text{Rad } R)[x]$,where R is an arbitrary ring, Rad $R[x]$ is the prime radical of the polynomial ring $R[x]$, and $(\text{Rad } R)[x]$ is the ideal of polynomials in the indeterminate x with coefficients in the prime radical Rad R of the ring R. The proof of this result has been given by McCoy [1]. The purpose of this note is to establish the same result above in the ring of power series in any finite number of indeterminates, with coefficients in a commutative Noetherian ring with identity. Throughout this note, R denotes a commutative ring with identity and $R[[x_1, \cdots, x_k]]$ is the ring of power series in the indeterminates $x_1, \cdots, x_k$, with coefficients in the ring R. Let us denote Rad $R[[x_k, \cdots, x_k]]$ the prime radical of $R[[x_1 \cdots, x^k]]$ and $(\text{Rad } R)[[x_1, \cdots, x_k]]$ the ideal of power series in the indeterminates $x_1, \cdots, x_k$, with coefficients in the prime radical Rad R of a ring R.

Lemma. If R is a Noetherian ring with identity, then Rad $R[[x]]=(\text{Rad } R)[[x]]$.

Proof. Let $\{P_i\}_{i\in I}$ be the collection of all prime ideals in R. If each P_i is a prime ideal of R then each $P_i[[x]]$ is a prime ideal of $R[[x]]$. Hence

$$\begin{aligned}\text{Rad}(R[[x]]) &\subseteq \cap_{i\in I} P_i[[x]] = (\cap_{i\in I} P_i)[[x]] \\ &= (\text{Rad } R)[[x]].\end{aligned}$$

To prove the reverse inclusion, let $f(x)$ be an arbitrary element of $(\text{Rad } R)[[x]]$. Let us set

$$f(x)=\textstyle\sum_{i=0}^{\infty} f_i x^i,$$

where the f_i are elements of Rad R. Then, since each f_i is in Rad R, it is nilpotent in R. Consider the ideal K generated by the coefficients of $f(x)$, that is,

$$K=(f_0, f_1, f_2, \cdots \quad).$$

Then

$$(f_0) \subseteq (f_0, f_1) \subseteq (f_0, f_1, f_2,) \subseteq \cdots.$$

Since R is a Noetherian ring, there exists a positive integer n such that

$$\begin{aligned}(f_0, f_1, \cdots, f_{n-1}) \subset (f_0, f_1, \cdots, f_n) &= (f_0, f_1, \cdots, f_{n+1}) \\ &= \cdots = K\end{aligned}$$

Since each f_i is nilpotent, there exist positive integers k_i such that $f_i^{k_i}=0$ for $i=0, 1, \cdots$. Let $k = k_0+k_1+\cdots+k_n$. Then, since each element of K is of the form:

$$g= r_0f_0+r_1f_1+\cdots +r_nf_n,$$

where $r_i\epsilon R, 0\leq i\leq n$, $g^k=0$. Thus each element of K is nilpotent, so that K is a nil ideal of R, and hence it is nilpotent, for R is a Noetherian ring. Therefore there exists some positive integer m such that $K^m=0$. This means that for every choice of m elements in K, their product is zero. On the other hand, since each coefficient of $[f(x)]^m$ is the sum of m elements in the set of coefficients of $f(x)$ and each coefficient of $f(x)$ is in K, each coefficient of $[f(x)]^m$ is zero, that is $[f(x)]^m=0$. Hence $f(x)\epsilon$ Rad $R[[x]]$.

Theorem. If R is a Noetherian ring with identity, then Rad $R[[x_1, \cdots, x_k]]=$ (Rad R) $[[x_1\cdots, x_k]]$.

Proof. The proof is by induction on k. For the case $k=1$, it is obvious by the Lemma. To complete the proof we must establish the Theorem for $k>1$ indeterminates under the assumption that it holds for $k-1$ indeterminates. That is, we assume that:

$$\text{Rad } R[[x_1, \cdots, x_{k-1}]]=(\text{Rad } R)[[x_1, \cdots, x_{k-1}]]$$

Since $R[[x_1, \cdots, x_{k-1}]]$ is also a Noetherian ring with identity and $R[[x_1, \cdots, x_k]]=R[[x_1, \cdots, x_{k-1}]][[x_k]]$,

$$\begin{aligned}\text{Rad } R[[x_1, \cdots, x_k]] &= \text{Rad } (R[[x_1, \cdots, x_{k-1}]] [[x_k]]) \\ &= (\text{Rad } R[[x_1, \cdots, x_{k-1}]])[[x_k]] \\ &= (\text{Rad } R)[[x_1, \cdots, x_{k-1}]][[k_k]] \\ &= (\text{Rad } R)[[x_1, \cdots, x_k]]\end{aligned}$$

References

1. McCoy, Neal H. The Prime Radical of a Polynomial Ring, Publ. Math. (Debrencen), 4, 1956.

2. Zariski, O. and Samuel, P. Commutative algebra. vol. I, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1958.

3. Burton, David M. A first course in rings and ideals, Addison Wesley series in Math. 1968.

4. Lambek, Joachim. Lectures on Rings and Modules, Waltham, Mass. Blaisdell, 1966.