Conformal Change in Einstein's
*\(g^{\mu\nu}\) -Unified Field Theory. -II, The Vector \(S_i\)

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Summary. —In the first paper of this series, [2], we investigated how the conformal change enforces the connections and gave the complete relations between connections in Einstein's *\(g^{\alpha\beta}\)-unified field theory. In the current paper we wish to investigate how the vector

\[
S_\mu^\nu = S_i
\]

is transformed by the conformal change. This topic will be studied for all classes and all possible indices of inertia.

1. Auxiliary Results

This section contains some known results taken from [1] and [2] without proofs, which are needed in our subsequent considerations. The same abbreviations, notations, and terminologies will be used.

A) Consider two space-times \(X_4\) and \(Y_4\), on which the differential geometric structure is imposed by the general real tensors \(*g^{\mu\nu}\) and \(*\tilde{g}^{\alpha\beta}\) respectively through the respective connections \(\Gamma_{\lambda \nu}^\mu\) and \(\tilde{\Gamma}_{\lambda \nu}^\mu\) defined by

\[
\begin{align*}
D_{\mu} *g^{\lambda \nu} &= -2 S_{\alpha \nu} *g_{\lambda \mu}, \\
D_{\mu} *\tilde{g}^{\lambda \nu} &= -2 S_{\alpha \nu} *\tilde{g}_{\lambda \mu}.
\end{align*}
\]

We say that \(X_4\) and \(Y_4\) are conformal if, and only if

\[
*\tilde{g}^{\lambda \nu}(x) = e^{-\Omega} *g^{\lambda \nu}(x),
\]

where \(\Omega = \Omega(x)\) is an arbitrary function of position with at least two derivatives. This conformal change enforces a change of connection, and it can always be expressed as follows ([2], p. 204):

\[
\begin{align*}
\tilde{\Gamma}_{\lambda \nu}^\mu &= \Gamma_{\lambda \nu}^\mu + Q_{\lambda \nu}^\mu + M_{\lambda \mu}^\nu + N_{\lambda \mu}^\nu,
\end{align*}
\]

where

\[
\begin{align*}
M_{\lambda \mu}^\nu &= -Q_{\lambda \mu}^\nu, \\
N_{\lambda \mu}^\nu &= Q_{\lambda \mu}^\nu.
\end{align*}
\]

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Throughout the present paper, Greek indices take values I, II, III, IV unless explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic components of a tensor and run from 1 to 4. Roman indices also follow the summation convention with the exception of indices \(x, y, z, t\).
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B) (1.1)a can be reduced to ([2], p. 205–206)

\[
M_{\alpha \beta \gamma} X_{\alpha \beta \gamma} = C_{\alpha \beta \gamma}
\]

where

\[
X_{\alpha \beta \gamma} \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 \\ A_{\alpha \beta} & A_{\alpha \gamma} & A_{\alpha \delta} + 2 A_{\alpha \beta \gamma} \\ A_{\alpha \beta \gamma} & A_{\alpha \gamma \delta} & A_{\alpha \beta \delta} \end{pmatrix}
\]

\[
C_{\alpha \beta \gamma} \overset{\text{def}}{=} \frac{1}{2} \left( H_{\alpha \beta \gamma} + 3 H_{\alpha \beta \gamma \delta} \right)
\]

\[
H_{\alpha \beta \gamma} \overset{\text{def}}{=} Q_\alpha k_{\beta \gamma} + 2Q_\alpha k_{\beta \gamma \delta} + h_{\delta \alpha \beta \gamma}
\]

\[
Q_\alpha \overset{\text{def}}{=} \partial_\alpha \Omega
\]

For the last two classes, (1.4)a is equivalent to ([1], p. 210)

\[
2M_{\alpha \beta \gamma} = H_{\alpha \beta \gamma} + Q_\alpha \frac{2}{3} k_{\beta \gamma} * k_{\beta \gamma} \text{ for the third class, and}
2M_{\alpha \beta \gamma} = Q_a k_{\beta \gamma} + 2Q_a k_{\beta \gamma \delta} \text{ for the fourth class.}
\]

2. Conformal Change of the Vector \( S_\lambda \)

THEOREM (2.1)a. (For the first two classes). The nonholonomic components of

\[
M_i \overset{\text{def}}{=} M_{\lambda i}
\]

are

\[
\begin{align*}
(2.1)a & \quad M_i = M_{\alpha \beta a} + M_{\alpha e f} + M_{\alpha f e} \\
(2.1)b & \quad M_i = M_{e f} + M_{e a b} + M_{e b a}
\end{align*}
\]

\[\text{Proof. Using the matrix equation (2.6) ([1], p. 1301), we have}
\[
M_i = M_{\alpha i} = M_{\alpha b} + M_{\alpha e f} + M_{\alpha f e}
\]

\[= M_{\alpha i} \ast h^{\alpha} + M_{\alpha e} \ast h^e + M_{\alpha f e} \ast h^f
\]

\[= M_{\alpha b} + M_{\alpha e f} + M_{\alpha f e},
\]

The second relation may be obtained similarly.

THEOREM (2.1)b. (For the first two classes). The nonholonomic components of

\[
M_i \overset{\text{def}}{=} M_{\lambda i}
\]

are given by

\[
(2.2)a & \quad M_i = \lambda_i Q_i,
\]

which is equivalent to

\[
(2.2)b & \quad M_i = Q_i * k_i^\lambda.
\]

\[\text{Proof. The non-holonomic components of } H_{\alpha \beta \gamma} \text{ are given by}
\]

\[
H_{\lambda \beta \gamma} = \lambda \Omega_{\lambda \gamma}^\beta + \lambda \Omega_{\gamma \beta} \ast \lambda \Omega_{\lambda \beta} \ast \lambda \Omega_{\gamma \beta} \ast h_{\lambda \beta \gamma},
\]

which is equivalent to

\[
(2.3) \quad H_{\alpha \beta \gamma} = H_{\lambda \beta \gamma} = \lambda \Omega_{\lambda \beta} \ast \lambda \Omega_{\lambda \beta} \ast \lambda \Omega_{\lambda \beta} = H_{\alpha \beta \gamma} = \lambda \Omega_{\lambda \beta} \ast \lambda \Omega_{\lambda \beta} \ast \lambda \Omega_{\lambda \beta}.
\]

\[\text{This result may be obtained by substituting (2.6) ([1], p. 1301) into (1.4)d.}\]
Since the type of two equations (1.4)a and (4.16) ([1], p. 1311) are similar, the non-holonomic components of $M_{
u
u}$ are obtained from (5.2) ([1], p. 1313) as follows:

$$4 \star \lambda M_{x y z} = (2 + \star \gamma \star \lambda \star \lambda) H_{x y z} + \star \lambda (\star \gamma \lambda \star \lambda) H_{z x y} + \star \lambda (\star \gamma \lambda \star \lambda) H_{y x z}.$$  

Substituting (2.3) into above, we have

$$4M_{a b a} = -\frac{2}{\star \gamma \lambda} H_{a b a} = 0,$$

(2.4)

$$4M_{a f e} = \frac{1}{\star \gamma \lambda} (2 + \star \gamma \star \lambda \star \lambda) H_{a f e} + \star \lambda (\star \gamma \lambda \star \lambda) H_{f a e} + \star \lambda (\star \gamma \lambda \star \lambda) H_{f e a} = -2 \lambda \Omega_{a f e}.$$  

Hence, by (2.1)a and (2.4) we have $M_a = \star \lambda \Omega_a$. Similarly we may easily see that (2.2)a holds for the case $i = e$.  

**Theorem (2.2).** The vector $S_\lambda$ is transformed by the conformal change (1.2) as

(2.5)

$$S_\lambda = S_\lambda + \Omega_a \lambda \xi^a.$$  

**Proof.** We have from (1.3)

$$S_{a b} \xi^a = S_{a b} \xi^a + M_{a b} \xi^a$$

so that

$$S_\lambda = S_\lambda + M_\xi.$$  

Theorem (2.1)b shows that (2.5) holds for the first two classes. On the other hand, since $\star k_a = 0$ for the last two classes, we have from (1.4)d and (1.5)

$$2M_\xi = H_{a b} \xi^a + \alpha_{\xi} \star k_a = -\Omega_a \star k_a + 2\Omega_a k_a \xi^a \xi^b = 2\Omega_a \star k_a.$$  

Hence, even in the last two classes we see that (2.5) holds.

**References**
