GENERALIZATION OF DEVELOPABLE SPACES

IL HAE LEE

O. Introduction

In recent years there have been several generalizations of developable spaces. For example, Bennett [3] defined a quasi-developable space which was proved to be useful in obtaining metrization theorems for M-spaces and linearly orderable topological spaces. Alexander [1] introduced semi-developable spaces and proved that a space is semi-metrizable if and only if it is a semi-developable $T_0$-space. Also he introduced cushioned pair-semidevelopable spaces [2] to obtain a generalization of Morita's metrization theorem.

In fact, a development for a space has three conditions as shown in the definition 1.1. If we pick out one essential condition, we get a natural generalization of (quasi-/semi-) developments, which we call qs-developments. This paper concerns with qs-developable spaces and with cushioned pair qs-developable spaces. The main results are:

1. A space is semi-developable if and only if it is qs-developable and perfect.
2. Every separable regular $T_0$-space with a point-finite qs-development is metrizable.
3. A cushioned pair qs-developable $T_0$-space is a Nagata space.
4. The Sorgenfrey line is not qs-developable.
5. There exists a non-developable (and hence, non-metrizable) cushioned pair qs-developable $M_1$-space.

Nearly all topological terminologies and symbols appearing in this paper is consistent with that used in Kelly [13]. Exceptions on symbols are closure and interior of a set $A$, we denote them by $cl(A)$ and $Int(A)$, respectively. We also adopt the convention that if $\gamma$ is a collection of sets, then $\gamma^+$ denotes the union of all sets in $\gamma$, and $st(x, \gamma)$ denotes the union of all sets in $\gamma$ containing $x$.

1. Quasi-semi-developable spaces.

Let $\gamma=(\gamma_1, \gamma_2, \ldots)$ be a sequence of collections of subsets of a space $(X, \tau)$. Consider the following conditions on $\gamma$:

(a) for each $x \in X$, $\{st(x, \gamma_n) : n \in \mathbb{Z}^+ \text{ and } x \in \gamma_n^+\}$ is a local base at $x$,
(b) each $\gamma_n$ is a covering of $X$ and
(c) each $\gamma_n$ is a subclass of $\tau$.

The above condition (a) is equivalent to the followings:

1) For each $x \in X$ and for each positive integer $n$ such that $st(x, \gamma_n) \neq \emptyset$, $st(x, \gamma_n)$ is a neighborhood of $x$, and

Received by the editors Apr. 1, 1974.
2) For each \(x \in X\) and for each open \(U\) containing \(x\), there exists a positive integer \(n\) such that \(x \in \text{st}(x, \gamma_n) \subseteq U\).

**Definition 1.1.** \(\gamma\) is called a quasi-semi-development (for brevity, a qs-development) for \(X\) if \(\gamma\) satisfies the condition (a). A space is said to be quasi-semi-developable (qs-developable) if it has a qs-development.

Recall that \(\gamma\) is a semi-development [1] if it satisfies (a) and (b); a quasi-development [3] if it satisfies (a) and (c); a development [16] if it satisfies (a), (b) and (c).

From the definition it is clear that (semi/quasi-)developable spaces are qs-developable, and that qs-developable spaces are 1-st countable. As the examples 3.1 and 3.2 show, no converse is true.

As the case of quasi-developable spaces, the following is true for qs-developable spaces:

**Theorem 1.2.** A space is semi-developable if and only if it is qs-developable and perfect (i.e., every closed set is a \(G_\delta\)).

**Proof.** Let \(\gamma = (\gamma_1, \gamma_2, \ldots)\) be a qs-development for a space \(X\). For each \(x \in \gamma_i^*\), \(\text{st}(x, \gamma_i)\) is a neighborhood of \(x\). This implies that each \(\gamma_i^*\) is open. Since \(X\) is perfect, each \(\gamma_i^*\) is a \(F_\sigma\)-set. Let \(\gamma_i^* = \bigcup F_{ij}\), where each \(F_{ij}\) is closed. For each \(i\) and \(j\), let \(\delta_{ij} = \gamma_i \cup \{x - F_{ij}\}\). Now we show that \(\{\delta_{ij}\}\) is a desired semi-development.

\(\text{st}(x, \delta_{ij})\) is a neighborhood of \(x\) whenever it is non-empty. Now let \(R\) be a neighborhood of \(x\). There is an \(i\) such that \(x \in \text{st}(x, \gamma_i) \subseteq R\). \(\gamma_i^* = \bigcup F_{ij}\) implies that there is a \(j\) such that \(x\) is contained in \(F_{ij}\). It follows that \(\text{st}(x, \delta_{ij}) = \text{st}(x, \gamma_i)\), and hence, \(\text{st}(x, \delta_{ij})\) is a neighborhood of \(x\) contained in \(R\).

For the converse, note that semi-developable spaces are perfect.

**Definition 1.3 [10].** A space is said to be \(z_1\)-compact if every uncountable subset has a limit point.

**Definition 1.4.** A qs-development \(\gamma = (\gamma_1, \gamma_2, \ldots)\) is said to be point-finite (point-countable) if each \(\gamma_i^*\) is point-finite (point-countable).

It is generally known that Lindelöf Moore spaces and \(\aleph_1\)-compact Moore spaces are metrizable. Bennett [3] proved that in a quasi-developable space, hereditary \(\aleph_1\)-compactness, hereditary Lindelöf and hereditary separability are equivalent: and each of these conditions implies the metrizability of the space if it is regular. On the other hand, Alexander [1] showed that a separable regular \(T_\gamma\)-space with a point-finite semi-development is metrizable. In the case of qs-developable spaces, some modifications are needed.

**Lemma 1.5.** In a hereditarily \(\aleph_1\)-compact qs-developable space every uncountable subset contains a condensation point.

**Theorem 1.6.** In a qs-developable space each of the followings are equivalent:

1) hereditarily \(\aleph_1\)-compact,
2) hereditarily Lindelöf and
3) hereditarily separable.

Further, if the space is regular \(T_\gamma\) and has a point-finite qs-development, then each of these conditions insures the metrizability of the space.
The first assertion of the above theorem 1.6 is proved using a generalization of the proof for quasi-developable spaces ([3], Theorem 2.5). The remaining part of the theorem 1.6 is an easy consequence of the theorem 1.8.

Remark. In regular qs-developable $T_0$-spaces the above conditions 1), 2) and 3) of theorem 1.6 are not sufficient to insure the metrizability of the spaces as seen in [14]: Example 3.2 exhibits a paracompact hereditarily separable semi-metric space which is not quasi-developable. Such a space is non-metrizable. This property also distinguishes qs-developable spaces from quasi-developable spaces.

McAuley [4] gave an example of separable semi-metric space which is not hereditarily separable, and Alexander [1] proved that a separable $T_0$-space with a point-finite semi-development is hereditarily separable. We generalize these to qs-developable spaces. The following lemma can be proved by analogous method to the proposition 1.12 of [1].

Lemma 1.7. A separable space with a point-countable qs-development is hereditarily separable.

Theorem 1.8. A separable regular $T_0$-space with a point-finite qs-development has a point-finite semi-development, and hence is metrizable.

Proof. Such a space is hereditarily separable by the lemma 1.7, and hence is hereditarily Lindelöf by the theorem 1.6. It is not difficult to show that a regular hereditarily Lindelöf $T_0$-space is perfect. In the proof of the theorem 1.2, perfect space with a point-finite semi-development has a point-finite semi-development. Use the theorem 1.7 of [1].

In [17], the notions of $\theta$-base and $\theta$-refinability were introduced in order to characterize developable spaces. Recently, Bennett gave an excellent characterization of quasi-developable spaces by means of $\theta$-base: A space is quasi-developable if and only if it has a $\theta$-base ([4], Theorem 8). He also introduced the concept of weak $\theta$-refinability and showed that weak $\theta$-refinability is a sufficient condition for a perfect space to be subparacompact ([7]).

The proposition 7 of [4] can be generalized as follows:

Proposition 1.9. A qs-developable space is (hereditarily) weakly $\theta$-refinable.

Corollary 1.10. (McAuley) A semimetric space is $\theta$-refinable, and hence subparacompact.

Proof. A semi-metric space is qs-developable and perfect. By the proposition 1.9, it is weakly $\theta$-refinable. Use the theorem 5 of [4].

2. Cushioned pair qs-developable spaces.

We now introduce a new class of spaces, cushioned pair qs-developable spaces, which is a generalization of cushioned pair-semidevelopable spaces.

Definition 2.1. If $\gamma$ and $\delta$ are collections of subsets of $X$, then we say that $\gamma$ is cushioned in $\delta$ if one can assign to each $G \in \gamma$ a $D(G) \in \delta$ such that, for every $\gamma' \subseteq \gamma$,

$$\alpha(\bigcup \{G: G \in \gamma'\}) \subseteq \bigcup \{D(G): G \in \gamma'\}.$$

By a cushioned pair semi-development (qs-development) we shall mean a pair of semi-
development (qs-development) \((\gamma, \delta)\) such that \(\gamma_n\) is cushioned in \(\delta_n\) for each \(n\), and such that

\[(*) \quad \gamma_1^* \subseteq \gamma_2^* \subseteq \gamma_3^* \subseteq \cdots\]

The above definition of a cushioned pair semi-development is due to Alexander [2]. In the definition of cushioned pair qs-developments, the condition that

\[(*) \quad \gamma_1^* \subseteq \gamma_2^* \subseteq \gamma_3^* \subseteq \cdots\]

is essential. Let \(X\) be a qs-developable space with a qs-development \(\gamma = (\gamma_1, \gamma_2, \cdots)\). If we set

\[
\delta_{2n} = \gamma_n, \quad \delta_{2n-1} = \{\phi\}
\]

and

\[
\eta_{2n} = \{X\}, \quad \eta_{2n-1} = \gamma_n
\]

then \((\delta, \eta)\) is a cushioned pair qs-development without the condition \((*)\).

Clearly a cushioned pair semi-development is a cushioned pair qs-development. That a space has a cushioned pair semi-development is, however, a very strong condition. In fact, we have

**Theorem 2.2** (Alexander) A space is metrizable if and only if it is \(T_0\) and has a cushioned pair semi-development.

The following theorem and example 3.3 show that cushioned pair qs-developable spaces locate between metrizable spaces and stratifiable spaces (=M3-spaces, see [6]) and they also distinguish cushioned pair qs-developable spaces from cushioned pair semi-developable spaces.

**Theorem 2.3.** A cushioned pair qs-developable \(T_0\)-space is a Nagata spaces (=a 1st countable space).

**Proof.** Let \((\gamma, \delta)\) be a cushioned pair qs-development for a space \(X\). We may assume that the set of all isolated points of \(X\) is contained in \(\gamma_1^*\). Let

\[
\mathcal{D}_n = \{(\text{Int st}(x, \gamma_n), \text{st}(x, \delta_n)) : x \in \gamma_n^*\}
\]

for each \(n\). Then each \(\mathcal{D}_n\) is a cushioned collection. To show that \(\mathcal{D} = \bigcup \mathcal{D}_n\) is a pair-base, let \(U\) be a neighborhood of a point \(x\). If \(x\) is an isolated point of \(X\), choose an \(n\) such that \(x \in \text{st}(x, \delta_n) \subseteq U\). Evidently \(x \in \text{Int st}(x, \gamma_n) \subseteq \text{st}(x, \delta_n) \subseteq U\). When \(x\) is not isolated, choose an \(m\) such that \(x \in \gamma_m^*\). \(\cap \{\text{st}(x, \delta_k) : x \in \delta_k^* \text{ and } k < m\} \subseteq U\) is a neighborhood of \(x\) with cardinality \(\geq 2\). Since \(\delta\) is a qs-development for \(X\), there is an \(n\) such that \(x \in \text{st}(x, \delta_n) \subseteq \bigcup \{\text{st}(x, \delta_k) : x \in \delta_k^* \text{ and } k < m\} \subseteq U\). For this \(n\), we have \(x \in \text{Int st}(x, \gamma_n) \subseteq \text{st}(x, \delta_n) \subseteq U\).

It is not so difficult to show that a cushioned pair qs-developable \(T_0\)-space is \(T_1\).

**Remark.** A qs-developable \(T_0\)-space need not be \(T_1\) as the Sierpinski space shows: \(X = \{a, \delta\}\) with the topology \(\phi, \{a\}\) and \(X\).

**Corollary 2.4.** A cushioned pair qs-developable \(T_0\)-space is semi-developable, and hence is semi-metrizable.

**Proof.** By the above theorem, a cushioned pair qs-developable \(T_0\)-space is stratifiable. A stratifiable space is perfect ([9], Theorem 2.2), and a perfect qs-developable space is semi-developable by the theorem 1.2. Recall that a semi-developable \(T_0\)-space is semi-


3. Examples

Example 3.1. There exists a qs-developable space which is neither quasi-developable nor semi-developable. Let $X$ be the space of real line with the topology: Each irrational point is isolated and open intervals with rational endpoints are open. Let $Y$ be the bow-tie region space of Heath. Let $Z$ be the topological sum of $X$ and $Y$. Clearly $Z$ is qs-developable. Since semi- (or quasi-) developability is a hereditary property, $Z$ is neither quasi-developable nor semi-developable.

Consider the Sorgenfrey line (half open interval space, $[13]$, 1K).

Assume there is a qs-development $\gamma=(\gamma_1, \gamma_2, \ldots)$ for the Sorgenfrey line $X$. Since $\{st(x, \gamma_n): n \in \mathbb{Z}^+ \text{ and } x \in \gamma_n^*\}$ is a local base at $x$, for each $x$ in $X$, there are positive integer $n(x)$ and a positive number $\varepsilon(x)$ such that $[x, x+\varepsilon(x)) \subseteq st(x, \gamma_n(x)) \subseteq [x, \infty)$.

Let $X_k = \{x \in X: n(x) = k\}$. Clearly $\bigcup X_k = X$. Since $X$ is uncountable, there is an $n$ such that $X_n$ is uncountable. Let $x$ and $y$ be distinct points of $X_n$. We may assume $x < y$. If $[x, x+\varepsilon(x)) \cap [y, y+\varepsilon(y)) \neq \emptyset$,

then $y \in [x, x+\varepsilon(x)) \subseteq st(x, \gamma_n)$. This implies that $x \in st(y, \gamma_n)$.

But $st(y, \gamma_n) \subseteq [y, \infty)$ and $x \in st(y, \gamma_n)$ with $x < y$ are incompatible. This contradiction shows that $\{[x, x+\varepsilon(x)): x \in X_n\}$ is a disjoint collection of uncountably many intervals, which is impossible. This completes the proof that the Sorgenfrey line is not qs-developable.

Example 3.3. There exists a non-developable (and hence, non-metrizable) cushioned pair qs-developable $M_1$-space. See $[9]$ for the definition of $M_1$-space.

Let $R'$ be the rational numbers. For $x \in R$, put $L_x = \{(x, y): (x, y) \in R \times R, 0 < y\}$ and $X = R \cup (R \times R)$. Then we will define a base for $X$ as follows: For $s, t \in R'$ and $r, s, t \in R$ such that $s < z < t$, we put $U_{s, t}(z) = (s, t)$ and let $\mathcal{U}$ be the set of all such $U_{s, t}(z)$. For $r, s, t \in R'$ and $z \in R$ such that $s < z < t$ and $r > 0$, we put $V_{r, s, t}(z) = (s, t) \cup (\{(w, y): 0 < y < r, w \in (s, t) - \{z\}\})$, and let $\mathcal{G}$ be the set of all such $V_{r, s, t}(z)$. Now put $\mathcal{D} = \mathcal{U} \cup \mathcal{G}$.

Then $\mathcal{D}$ is a $\sigma$-closure preserving base making $X$ into a nonmetrizable first countable $M_1$-space.

Now we will define a cushioned pair qs-development as follows:

For each $n \in \mathbb{Z}^+$, $p \in R$ and $a \in R$, let

$M_{n, p} = \{x, y) \in X: y = a(x - p), (x - p)^2 + y^2 < 1/n^3\}$,

$N_{n, p} = \{p, y) \in X: 0 < y < 1/n\}$. 

Put
\[
\gamma'_{s,t} = \{U_{s,t}(x) : x \in R\}, \quad \text{where } s' = \frac{2s + t}{3} \text{ and } t' = \frac{s + 2t}{3},
\]
\[
\delta'_{s,t} = \{U_{s,t}(x) : x \in R\},
\]
\[
\gamma'_{n} = \{M_{2n}p : p \in R, a \in R\} \cup \{N_{2n}p : p \in R\},
\]
\[
\delta'_{n} = \{M_{2n}p : p \in R, a \in R\} \cup \{N_{2n}p : p \in R\}.
\]
Now let \( \varphi \) be an 1-1 correspondence from \( Z^+ \) onto \( R' \times R' \). Put
\[
\gamma_{2n} = \gamma'_{\varphi(n)}, \quad \gamma_{2n-1} = \gamma'_{n}; \quad \delta_{2n} = \delta'_{\varphi(n)}, \quad \delta_{2n-1} = \delta'_{n}.
\]
We will define a cushioned pair qs-development \( (\xi, \eta) \):

Put \( \xi_1 = \gamma_1 \). Assume \( \xi_{n-1} \) is defined. If \( \xi_{n-1} \times \gamma_{n} = \varnothing \), let \( \xi_n = \xi_{n-1} \cup \gamma_n \). If \( \xi_{n-1} \times \gamma_{n} \neq \varnothing \), let \( \xi_n = \gamma_n \cup \{C - \gamma_n : C \subseteq \xi_{n-1}\} \cup \{C \cap cl(\gamma_n) : C \subseteq \xi_{n-1}\} \). Similarly we define \( \eta_1 = \delta_1 \). Assume \( \eta_{n-1} \) is defined. If \( \eta_{n-1} \times \delta_{n} = \varnothing \), let \( \eta_n = \eta_{n-1} \cup \delta_n \). If \( \eta_{n-1} \times \delta_{n} \neq \varnothing \), let \( \xi_n = \delta_n \cup \{D - \delta_n : D \subseteq \eta_{n-1}\} \cup \{D \cap cl(\delta_n) : D \subseteq \eta_{n-1}\} \). One can verify that \( (\xi, \eta) \) is a cushioned pair qs-development for \( X \).

If \( X \) were developable, it would be metrizable since it is paracompact.

References


Seoul National University