SOME PROPERTIES OF PSEUDONORMABLE
SEMINILinear SPACES

BY WON HUH

1. Introduction.

F. F. Bonsall [1], [2] and S. Bourne [3], [4] have taken a semilinear algebras to be a subset of a Banach algebra which is closed under addition, multiplication and scalar multiplication by non-negative reals. And R. E. Worth [10] has defined the topological semilinear space is a semilinear space $S$ over $\mathbb{R}^+$, the set of all non-negative reals, with a Hausdorff topology such that addition and scalar multiplication are continuous. When a space is a normed linear space over a scalar field a topology on the space is defined by its norm or by its invariant metric. However this is not the case for a semilinear space, for the pseudonorm does not define an invariant pseudo-metric on the space. But we confine our study to a semilinear space with a pseudo-norm.

In this paper, we, with a topological structure of a semilinear space whose topology is not assumed to be Hausdorff, study a part of the classical theory of pseudonormable semilinear spaces. § 2 is a summary of relevant concepts and theorems which are employed in a part of the remainder of the paper. In § 3 the general theory is developed. In § 4 the quotient semilinear spaces are developed.

2. Preliminaries.

In this paper we shall use the terms in the sense given by the author in previous paper [13]. For the sake of completeness we repeat:

DEFINITION 1. A Half-field is a system consisting of a set $H$ and two binary operations called addition and multiplication with the following properties:

1. $H$ is a semi-group with identity (0) under addition.
2. $H$ with non-zero elements forms a commutative group under the multiplication.
3. The left-hand and right-hand distributive laws:

$$a(b+c)=ab+ac \quad \text{and} \quad (a+b)c=ac+bc$$

hold for every elements $a, b,$ and $c$ of $H$.

DEFINITION 2. A topological half-field is a half-field $H$ together with a topology on $H$ under which the half-field operations are continuous.

DEFINITION 3. A semilinear space over a scalar half-field $\mathbb{R}^+$ is a system of a set $E$ and two operations of addition and scalar multiplication by $\mathbb{R}^+$ with the following properties:

1. $E$ is a commutative and cancellative semigroup under the addition with the additive identity zero (0).
2. For every $x$ and $y$ of $E$ and for every $\alpha$ of $\mathbb{R}^+$, $\alpha(x+y)=\alpha x+\alpha y$.
3. For every $x$ of $E$ and for every $\alpha$ and $\beta$ of $\mathbb{R}^+$.
\[(a + b)x = ax + bx \quad \text{and} \quad \alpha(\beta x) = (\alpha\beta)x.\]

(4) For every \(x\) of \(E\) and for every \(\alpha\) of \(R^+\), \(ax = xa\).

(5) For every \(x\) of \(E\) \(1x = x\) and \(0x = 0\) where the zero of left-hand is the zero of scalar and that of right-hand is the zero of \(E\).

**Definition 4.** A subset of a semilinear space over scalar half-field \(R^+\) is said to be a semilinear subspace of the space if the addition and scalar multiplication operations are closed in the set.

**Definition 5.** A subset \(A\) of a semilinear space \(E\) over a scalar half-field \(R^+\) is said to be convex if for every pair of \(x\) and \(y\) of \(A\) and for every pair of \(\alpha\) and \(\beta\) of \(R^+\) with \(\alpha + \beta = 1\), \(ax + \beta y \in A\).

**Definition 6.** A pseudo-metric for a semilinear space \(E\) over a scalar half-field \(R^+\) is a non-negative real valued function \(d\) such that:

1. \(d(x, x) = 0\) for every \(x\) of \(E\),
2. For every \(x, y,\) and \(z\) of \(E\), \(d(x, y) \leq d(x, z) + d(y, z)\).

When a pseudo-metric \(d\) satisfies that \(d(x, y) = 0\) implies \(x = y\) we say that \(d\) is a metric for \(E\).

Here we remark that:

1. For every \(x\) and \(y\) of \(E\), \(d(x, y) = d(y, x)\).
2. Every metric is a pseudo-metric.

**Definition 7.** A pseudo-metric (metric) \(d\) for a semilinear space over a scalar half-field \(R^+\) is said to be pseudo-metric invariant (metric invariant resp.) if for every \(x, y,\) and \(z\) of \(E\) and for every \(\alpha\) of \(R^+\), \(d(ax + y, ax + y) = \alpha d(x, z)\).

**Definition 8.** A pseudo-norm on a semilinear space \(E\) over a scalar half-field \(R^+\) is a non-negative real valued function \(\nu\), defined for each point of \(E\), such that:

1. For each pair of points of \(E\), \(\nu(x + y) \leq \nu(x) + \nu(y)\).
2. For every \(\alpha\) of \(R^+\) and for every \(x\) of \(E\), \(\nu(\alpha x) = \alpha \nu(x)\).

When a pseudo-norm \(\nu\) satisfies that \(\nu(x) = 0\) implies \(x = 0\) we say that \(\nu\) is a norm.

**Theorem 1.** In a semilinear space \(E\) over a scalar half-field \(R^+\) an invariant pseudo-metric (invariant metric) defines a pseudo-norm (norm resp.).

**Proof.** Let \(d\) be an invariant metric and we define \(\nu(x) = d(x, 0)\) for every \(x\) of \(E\). Then we have

\[\nu(x + y) = d(x + y, 0) = d(x + y, 0) + d(0, 0) = d(x, 0) + d(y, 0) = \nu(x) + \nu(y),\]

and

\[\nu(\alpha x) = d(\alpha x, 0) = \alpha d(x, 0) = \alpha \nu(x)\]

for every \(\alpha\) of \(R^+\) and for every \(x\) and \(y\) of \(E\). If \(d\) is an invariant metric, then \(\nu(x) = d(x, 0) = 0\) implies \(x = 0\). This completes the proof.

**Definition 9.** A topological semilinear space is a semilinear space \(E\) over a scalar half-field \(R^+\) together with a topology in \(E\) such that: addition : \(E \times E \rightarrow E\) and scalar multiplication : \(R^+ \times E \rightarrow E\) are continuous with respect to the topology in \(E\).
Some properties of pseudonormable semilinear spaces

and the product topologies in $E \times E$ and in $R^+ \times E$.

Here we remark that:

1. The continuity of addition is that for any $a$ and $b$ of $E$ and for any neighborhood $U$ of $a+b$ there exist a neighborhood $V$ of $a$ and a neighborhood $W$ of $b$ such that $V+W \subseteq U$, and that of multiplication is that for any $a$ of $E$ and for any $\alpha$ of $R^+$ and for any neighborhood $U$ of $aa$ there exist a neighborhood $A$ of $\alpha$ and a neighborhood $V$ of $a$ such that $AV \subseteq U$.

2. Any neighborhood of $x$ of $E$ is denoted by $x+V$ where $V$ is a neighborhood of the origin $0$.

**Definition 10.** A topological semilinear space is locally convex if for every neighborhood $U$ of the origin there exists a convex neighborhood $V$ of the origin such that $V \subseteq U$.

**Definition 11.** A subset $B$ of a topological semilinear space is bounded if every neighborhood $U$ of the origin there exists a positive integer $n$ such that $B \subseteq nU$.

**Definition 12.** A topological semilinear space is locally bounded if there exists a bounded neighborhood of the origin.

**Theorem 2.** Every locally bounded topological semilinear space $E$ is a first countable space.

**Proof.** It is sufficient to prove that it has a countable base at the origin. By definition $E$ has a bounded neighborhood $U$ of the origin. Let $\mu_\alpha : E \rightarrow E$ be a function such that $\mu_\alpha(x) = \alpha x$ for every $x$ of $E$ and for every $\alpha$ of positive real numbers. Since the scalar multiplication is continuous, so is $\mu_\alpha$. By the identity $\mu_\alpha \circ \mu_\alpha^{-1} = \mu_\alpha^{-1} \circ \mu_\alpha = I_E$, we have $\mu_\alpha$ is a homeomorphism. Furthermore, $\mu_\alpha(U) = \{\alpha x | x \in U\} = \alpha U$ is a neighborhood of the origin for every $\alpha$ of the positive real numbers. Let $V$ be any neighborhood of the origin. Then there exists a positive integer $n$ such that $U \subseteq nV$, that is, $\frac{1}{n}U \subseteq V$, hence $\{\frac{1}{n}U | n \in N\}$ forms a countable base at the origin. This completes the proof.

3. Pseudonormable semilinear spaces.

In a linear space $E$ over $R^+$ if $\nu$ and $d$ are pseudo-norm and an invariant pseudo-metric for $E$ respectively, then they are definable each other by $d(\alpha x, \beta y) = \nu(\alpha x - \beta y)$ for every $x$ and $y$ of $E$ and for every $\alpha, \beta$ of $R^+$.

**Theorem 3.** Every semi-linear space $E$ over a scalar half-field $R^+$ with an invariant pseudo-metric is a topological semilinear space with the invariant pseudo-metric topology, more precisely a locally convex and locally bounded (topological) semilinear space.

**Proof.** By Theorem 1 since

$$d(x+y, a+b) = d(x+y, y+a) + d(y+a, a+b) = d(x, a) + d(y, b),$$

the addition is continuous. Let $\beta$ be any element of a neighborhood of a scalar $\alpha$ and $y$ any element of neighborhood of an $x$ of the space, then, since $\alpha \geq \beta$ implies that there
exists an $\varepsilon > 0$ such that $\alpha = \beta + \varepsilon$, we have
\[
d(\alpha x, \beta y) \leq d(\alpha x, \alpha y) + d(\alpha y, \beta y)
= \alpha d(x, y) + d(\beta y + \varepsilon y, \beta y) = \alpha d(x, y) + \varepsilon d(y, 0),
\]
which proves that the scalar multiplication is continuous. Let $B_\rho = \{x \mid d(x, 0) < \rho\}$ be any neighborhood of the origin and $y, z \in B_\rho$ and $\alpha + \beta = 1$. Then we have, by setting $\nu(x) = d(x, 0)$, $\nu(\alpha y + \beta z) \leq \alpha \nu(y) + \beta \nu(z) < \alpha \rho + \beta \rho = \rho$, that is, $\alpha y + \beta z \in B_\rho$, which shows the local convexity of the space. Let $U$ be any neighborhood of the origin, Then there exists a $B_\rho = \{x \mid d(x, 0) < \rho\} \subseteq U$. Hence $B_\frac{1}{\rho} = B_\rho \subseteq B_1 = U$, and since there exists a positive integer $n$ such that $\frac{1}{\rho} < n$, we have $B_1 \subseteq n U$, which shows the local boundedness of the space. This completes the proof.

**DEFINITION 13.** A topological semilinear space defined by a pseudo-norm (norm) is called a pseudonormed (normed) semilinear space.

We remark that $\nu$, defined by $\nu(x) = d(x, 0)$, is a pseudonorm (norm resp.) for the space.

**DEFINITION 14.** A topological semilinear space is pseudonormable (normable) if and only if there exists a pseudo-norm (norm resp.) whose topology is that of the space.

We remark that the topology of a normable semilinear space is Hausdorff, and that any normable topological semilinear space is pseudonormable.

**LEMMA 1.** In a topological semilinear space $E$ with the topology $\mathcal{T}$ if a subset $B$ of $E$ is convex then so is the interior $\text{Int}(B)$ of $B$.

**Proof.** Since $\text{Int}(B) = \bigcup \{U \in \mathcal{T} \mid U \subseteq B\}$, for any $x$ and $y$ of $\text{Int}(B)$ there exist a neighborhood $U$ of $x$ and a neighborhood $V$ of $y$ such that $U \subseteq B$ and $V \subseteq B$. Since for $\alpha$ and $\beta$ with $\alpha + \beta = 1$ in $R^+$, $\alpha U + \beta V \subseteq \alpha B + \beta B \subseteq B$, $\alpha x + \beta y \in \text{Int}(B)$.

**THEOREM 4.** A topological semilinear space $E$ is pseudonormable if and only if it is locally convex and locally bounded.

**Proof.** “Only if” is obvious by Theorem 3. “If”: Assume that the condition holds. By the remark of Definition 9 and by Theorem 3 it is sufficient to consider at the origin. By Lemma 1 there exists a bounded convex neighborhood $V$ of the origin, and clearly there exists an $\alpha \in R^+$ such that $x \in \alpha V$ for every $x$ of $E$. Let $\sigma$ be a relation such that $x \sigma y$ if and only if there exist an $\alpha \in R^+$ and $v \in \alpha V$ such that $x + v = y + v$. It is easy to check that $\sigma$ is an equivalence relation. We denote $\nu(x) = \nu(y)$ for each pair of $x$ and $y$ of $E$ which satisfies $x \sigma y$. Let’s define $\nu(x) = \inf \{\rho \in R^+ \mid x \in \rho V\}$.

Then for any $x$ and $y$ of $E$ and for arbitrary $\varepsilon > 0$, $x \in (\nu(x) + \varepsilon)V$ and $y \in (\nu(y) + \varepsilon)V$.

Since $V$ is convex,
\[
\frac{\alpha x}{\nu(x) + \varepsilon} + \frac{\beta y}{\nu(y) + \varepsilon} \in V \text{ for } \alpha + \beta = 1 \text{ in } R^+.
\]
Some properties of pseudonormable semilinear spaces

By setting $\alpha = -\frac{\nu(x) + \varepsilon}{\nu(x) + \nu(y) + 2\varepsilon}$ we have $\beta = -\frac{\nu(y) + \varepsilon}{\nu(x) + \nu(y) + 2\varepsilon}$ hence $x + y \in (\nu(x) + \nu(y) + 2\varepsilon) V$, that is, $\nu(x + y) \leq \nu(x) + \nu(y)$. By definition of $\nu$ we have, for any $\alpha \in \mathbb{R}^+$, $\nu(\alpha x) = \alpha \nu(x)$. This shows that $\nu$ is a pseudonorm for $E$.

Let $\mathcal{T}$ be the given topology and $\xi$ a topology for $E$ defined by the pseudo-norm $\nu$. We assert that $\xi = \mathcal{T}$.

Let $G \in \mathcal{T}$ be any neighborhood of the origin. By the boundedness of $V$, there exists a positive integer $n$ such that $\frac{1}{n} V \subseteq G$ and $\{x | \nu(x) < \frac{1}{n} \} \subseteq \frac{1}{n} V \subseteq G$, hence $G \in \xi$, that is, $\mathcal{T} \subseteq \xi$. Let $H$ be a neighborhood of the origin with respect to $\xi$. Then there exists a $\rho \in \mathbb{R}^+$ such that $\{x | \nu(x) < \rho \} \subseteq H$. Since there exists a $\delta \in \mathbb{R}^+$ such that $\nu(x) < \delta < \rho$, we have $\delta V \subseteq \{x | \nu(x) < \rho \} \subseteq H$, and $\delta V$ being open with respect to $\mathcal{T}$, $H \in \mathcal{T}$, that is, $\xi \subseteq \mathcal{T}$. This completes the proof.

By Theorem 2 and Theorem 4 we have the following

**Corollary.** Every pseudonormable semilinear space is first countable.

We remark that, since the property of a space being first countable is hereditary, any subspace of the pseudonormable semilinear space is first countable.

**Definition 15.** A non-empty subset $B$ of a directed set $D$ with a binary relation $\sigma$ which directs $D$ is residual if there exists an element $d$ of $D$ such that $d \sigma a$ implies $a \in B$.

We note that $T_d = \{a \in D | d \sigma a\}$ is residual.

**Definition 16.** A net in a topological semilinear space $E$ is a function $\varphi : D \to E$ of some directed set $D$, and we say that $\varphi$ converges to $a$ (denoted by $\varphi \to a$) if for every neighborhood $U$ of $a$ there exists a residual set $T_d$ such that $\varphi(T_d) \subseteq U$, and a net $\varphi : N \to E$ is called a sequence in $E$.

**Definition 17.** A closed subspace of a pseudonormed (normed) semilinear space $E$ is a topological semilinear subspace of the semilinear space $E$ which is a closed subset of the pseudonormed (normed resp.) semilinear space $E$.

**Theorem 5.** The closure $\text{cl}(X)$ of a semilinear subspace $X$ of a pseudonormable (normable) semilinear space $E$ is a convex subspace of $E$.

**Proof.** Since $E$ is first countable, for every $x$ of $\text{cl}(X)$ there exists a decreasing countable base $\{U_j | j \in N\}$ containing $x$. Since $x \in \text{cl}(X)$ if and only if there exists a net $\varphi$ in $X$ which converges to $x$ (See, [7]), $x \in \text{cl}(X)$ if and only if there exists a directed set $D$ such that $\varphi : D \to X$ which converges to $x$, that is, for every $U_j$ there exists a residual set $T_d$ such that $\varphi(T_d) \subseteq U_j$. Choose $x_j \in \varphi(T_d) \subseteq U_j$ $(j \in N)$. Hence, by setting $\varphi(j) = x_j$ $\varphi$ is a sequence in $X$ with $\varphi \to x$, which shows that $x \in \text{cl}(X)$ if and only if there exists a sequence $\varphi$ in $X$ converging to $x$. Let $a \in \text{cl}(X)$ and $b \in \text{cl}(X)$. Then there exist sequences $f$ and $g$ in $X$ which converges to $a$ and $b$ respectively, hence sequences $f + g$ and $\alpha f$ $(\alpha \in \mathbb{R}^+)$ in $X$ converges to $a + b \in \text{cl}(X)$ and $\alpha a \in \text{cl}(X)$ respectively. This completes the proof.
Since any pseudonormed (normed) semilinear space is pseudonormable (normable resp.), we have the following

**Corollary.** The closure $cl(X)$ of a pseudonormed (normed) semilinear subspace $X$ of the space $E$ is a convex subspace.

**Theorem 6.** If a pseudonormable semilinear space $E$ is a $T_1$ space then $E$ is a normable semilinear space, and vice versa.

**Proof.** Let $\nu$ be a defining pseudonorm for $E$, and $x \neq 0$ in $E$. Since $E$ is $T_1$, $\{x\}$ is closed. By setting $E \setminus \{x\} = G$, $G$ is open and contains 0. Hence there exists a locally convex and locally bounded neighborhood $V$ of 0 such that $x \in V$. Hence $\nu(x) \neq 0$. This shows that $\nu$ is a norm for $E$, thus $E$ is normable. Conversely, if $\nu$ is a defining norm for $E$, then $x \neq 0$ in $E$ implies that $\nu(x) \neq 0$. Hence for $H = E \setminus \{x\}$ there exists a neighborhood $V$ of 0 such that $y + V \subseteq H$ for any $y$ in $H$, thus $\{x\}$ is closed. This completes the proof.

Since the normability carries the Hausdorff topology we have the following

**Corollary.** A pseudonormable semilinear space is a $T_1$ space if and only if it is a Hausdorff semilinear space.

**4. Quotient semilinear spaces.**

Let $E$ be a semilinear spaces over $\mathbb{R}^+$ and $S$ a semilinear subspace of $E$ and $\sigma$ the relation defined by the statement: $a \sigma b$ if and only if there exist $s_1$ and $s_2$ of $S$ such that $a + s_1 = b + s_2$ which is denoted by $a \equiv b \mod S$. It is easy to check that $\sigma$ is an equivalence relation in $E$. We denote by $E/S$ the quotient set of $E$ modulo the equivalence relation $\sigma$ defined by $S$.

**Definition 18.** Let $(X, \mathcal{F})$ be a topological space and $Y$ any set and $p : X \rightarrow Y$ a function. The identification topology in $Y$ determined by $p$ is $\mathcal{F}_p = \{G \subseteq Y : p^{-1}(G) \in \mathcal{F}\}$.

We remark that $\mathcal{F}_p$ is the largest topology in $Y$ for which $p : X \rightarrow Y$ is continuous.

**Theorem 7.** Let $E$ be a pseudonormable semilinear space and $S$ a semilinear subspace of $E$, $p : E \twoheadrightarrow E/S$ a canonical projection where $E/S$ has an identification topology. Then $p$ is a continuous linear function and $E/S$ is pseudonormable.

The space $E/S$ is called the quotient pseudonormable semilinear space of $E$ (modulo the equivalence relation defined) by a subspace $S$.

**Proof.** Since $E/S$ has an identification topology, the continuity of $p$ is clear (See, for example, [5]). Linearity of $p$: Let $a \in \mathbb{R}^+$ and $x, y \in E$. Then

\[
p(x) + p(y) = \{a | a \in p^{-1}(p(x)) \} + \{b | b \in p^{-1}(p(y)) \} = \{a + b | a \in p^{-1}(p(x)), b \in p^{-1}(p(y)) \} = \{a + b | a + b \equiv p^{-1}(p(x) + y) \} = p(x + y),
\]

\[
\alpha p(x) = \{a | a \equiv p^{-1}(p(x)) \} = \{a | a \equiv x \mod S \} = \{a | a \equiv ax \mod S \} = \{a | a \equiv ax \mod S \} = \{a | a \equiv ax \mod S \} = p(ax),
\]

which shows $p$ is linear. With the linearity of $p$ it is easy to see the semilinearity of
Construction of a pseudo-norm which defines the identification topology: Let's define $\nu^*$ as follows:
\[ \nu^*(p(x)) = \inf \{ \nu(a) \mid a \in p^{-1}p(x) \} \]
where $\nu$ is the pseudonorm for $E$ which defines the topology of the space to be pseudonormed. It is clear that $\nu^*$ is well-defined and $\nu^*(p(x)) \geq 0$ for every $p(x)$ of $E/S$. Let $\varepsilon > 0$ be arbitrary, then there exist an $a \in p^{-1}p(x)$ and a $b \in p^{-1}p(y)$ such that $\nu^*(p(x)) + \frac{1}{2}\varepsilon > \nu(a)$ and $\nu^*(p(y)) + \frac{1}{2}\varepsilon > \nu(b)$, and hence $\nu^*(p(x)) + \nu^*(p(y)) + \varepsilon > \nu(a) + \nu(b) \geq \inf \{ \nu(a+b) \mid a \in p^{-1}p(x), b \in p^{-1}p(y) \} = \nu^*(p(x) + p(y))$, since $\varepsilon > 0$ is arbitrary we have $\nu^*(p(x) + p(y)) \leq \nu^*(p(x)) + \nu^*(p(y))$, and for $a \in R^+$, $\nu^*(\alpha p(x)) = \alpha \nu^*(p(x))$ is obvious by the definition of $\nu^*$. Thus $\nu^*$ is a pseudonorm for $E/S$. We now prove that the topology defined by $\nu^*$ is the identification topology in $E/S$. Let $U$ be any neighborhood of the origin in $E/S$ with respect to the identification topology. Then $p^{-1}(U)$ is a neighborhood of the origin in $E$. Hence there exists a convex and bounded neighborhood $V$ of the origin in $E$ such that $V \subseteq p^{-1}(U)$, and there exists a $\rho \in R^+$ such that $V = \{ x \mid \nu(x) < \rho \}$. Then since $x \in p^{-1}(U)$ implies that $p(x) \equiv U$, $\nu(x) < \rho$ implies that $\{ p(a) \mid a \equiv V \} \subseteq U$. This shows that $U$ is also contained in the topology defined by $\nu^*$. This completes the proof.

By Theorem 6 we have the following

**Corollary.** The quotient pseudonormable semilinear space $E/S$ of a pseudonormable semilinear space $E$ modulo a semilinear subspace $S$ is Hausdorff if and only if $S$ is closed.

**Proof.** Necessity: Since $E/S$ is Hausdorff [0] in $E/S$ is closed. By the continuity of the canonical projection $p : E \rightarrow E/S$ $p^{-1}(0) = S$ is closed in $E$.

Sufficiency: Straightforward from Theorem 6.

**Theorem 8.** Let $E_j$ $(j = 1, 2)$ be pseudonormable semilinear spaces, and as binary operations in the product $E_1 \times E_2$ addition: $(a, b) + (x, y) = (a + x, b + y)$, scalar multiplication: $\alpha (a, b) = (\alpha a, \alpha b)$ are given, and let $d_j$ $(j = 1, 2)$ be pseudo-norms which define the topologies of $E_j$ respectively. Then the function $d_1 \times d_2 : (a, b) \rightarrow d_1(a) + d_2(b)$ is a pseudo-norm for the product space $E_1 \times E_2$ which defines the topology of $E_1 \times E_2$, hence $E_1 \times E_2$ is a pseudonormable semilinear space.

**Proof.** The semilinearity of the product $E_1 \times E_2$ is the consequence of the binary operations given. By definition of $d_1 \times d_2$,
1. $d_1 \times d_2(a, b) = d_1(a) + d_2(b) \geq 0$ for every element $(a, b)$ of $E_1 \times E_2$.
2. $d_1 \times d_2[(a, b) + (x, y)] = d_1 \times d_2(a + x, b + y) = d_1(a + x) + d_2(b + y) 
\leq d_1(a) + d_1(x) + d_2(b) + d_1(y) = d_1 \times d_2(a, b) + d_1 \times d_2(x, y)$ for every pair of $(a, b)$ and $(x, y)$ of $E_1 \times E_2$.
3. $d_1 \times d_2(\alpha (a, b)) = d_1 \times d_2(\alpha a, \alpha b) = d_1(\alpha a) + d_2(\alpha b) = \alpha d_1(a) + \alpha d_2(b) = \alpha d_1 \times d_2(a, b)$
for every $\alpha$ of $R^+$ and for every $(a, b)$ of $E_1 \times E_2$.

This shows that $d_1 \times d_2$ is a pseudonorm for $E_1 \times E_2$. Let $U_1 \times U_2$ be any neighborhood of $(a_1, a_2) + (b_1, b_2)$.

Then $U_j (j = 1, 2)$ are neighborhoods of $a_j + b_j (j = 1, 2)$ respectively, so that there exist neighborhoods $V_j$ of $a_j$ and neighborhoods $W_j$ of $b_j$ such that $V_j + W_j \subseteq U_j (j = 1, 2)$ respectively. Hence we have $U_1 \times U_2 \supseteq (V_1 + W_1) \times (V_2 + W_2) \supseteq V_1 \times V_2 + W_1 \times W_2$.

Since $V_1 \times V_2$ and $W_1 \times W_2$ are neighborhoods of $(a_1, a_2)$ and $(b_1, b_2)$ respectively, the continuity of addition is proved.

The continuity of scalar multiplication can be proved in a similar way.

Let $V \times W$ be any neighborhood of $(x_1, x_2)$ with respect to the product topology. Then $P_{\tau(i)}(G_1 \times G_2) = G_j$ (where $P_{\tau(i)}$ is the projection onto $j$-th factor) implies that there exists a convex and bounded neighborhood $H_j$ of $x_j$ with respect to the topology defined by $d_1$ such that $H_j \subseteq G_j (j = 1, 2)$ respectively. This completes the proof.

Let $E$ be a pseudonormable semilinear space and $S = \{(x, x) | x \in E\}$ and $\sigma$ the relation defined by the statement: $(a, b) \sigma (c, d)$ if and only if there exist $(x, x)$ and $(y, y)$ of $S$ such that $(a, b) + (x, x) = (c, d) + (y, y)$. It is easy to check that $\sigma$ is an equivalence relation, that is, $(a, b) \equiv (c, d) \text{ mod } S$.

By Theorem 7 we have the following

**Corollary.** Let $E$ be a pseudonormable semilinear space and $S = \{(x, x) | x \in E\}$. If $p : E \times E \rightarrow E \times E / S$ is a canonical projection where $E \times E / S$ has an identification topology. Then $E \times E / S$ is the quotient pseudonormable semilinear space.

**References**


Busan National University