ON COMPLETE VECTOR LATTICES OF ORDER BOUNDED LINEAR MAPPINGS

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Introduction.

When \( E \) is a vector lattice and \( F \) is an order complete vector lattice, the set of all order bounded linear mappings from \( E \) into \( F \), which is denoted by \( \mathcal{L}(E,F) \), forms an order complete vector lattice. (cf. [1]) We tried to find out some properties of \( \mathcal{L}(E,F) \) and its subspaces.

1. Preliminaries.

When a real vector space \( E \) is equipped with a partial order \( \leq \) having the following properties: if \( x \leq y \), then \( x+z \leq y+z \) for any \( x, y \) and \( z \) of \( E \) and if \( \alpha \) is a positive real number, then \( x \leq y \) implies \( \alpha x \leq \alpha y \), we call \( E \) an ordered vector space. We shall denote by \( E^+ \), the set \( \{ x \in E | x \geq 0 \} \). An ordered vector space \( E \) is called a vector lattice if there exist the supremum \( x \lor y \) and the infimum \( x \land y \) for every pair of elements of \( E \). Suppose that \( E \) is a vector lattice and \( x \in E \). We define \( x^+ = x \lor 0 \), \( x^- = (-x)^+ \) and call them the positive part and the negative part, respectively, of \( x \). The absolute value \( |x| \) of \( x \) is defined as \( x \lor (-x) \). It is easy to see that \( x = x^+ - x^- \); hence \( E = E^+ - E^- \) and that \( |x| = x^+ + x^- \). Let \( E \) be a vector lattice. A subset \( S \) of \( E \) is called order bounded if it is contained in some order interval \( [x,y] = \{ z \in E | x \leq z \leq y \} \). \( E \) is said to be order complete if every order bounded subset of \( E \) has the supremum and the infimum in \( E \). A net \( \{ x_\alpha \} \) is said to order converge to \( x \) in a vector lattice \( E \) if there exist an increasing net \( \{ y_\alpha \} \) and a decreasing net \( \{ z_\alpha \} \) such that \( \sup \{ y_\alpha \} = x = \inf \{ z_\alpha \} \) and for any \( x_\alpha \), there exist \( y_\beta \leq x_\alpha \leq z_\gamma \). A subset \( S \) of \( E \) is said to be closed if \( S \) contains the limit of every order convergent net in \( S \). Suppose that \( E \) is an ordered vector space and that \( E \) is the direct sum of two linear subspaces \( M \) and \( N \). \( E \) is called the order direct sum of \( M \) and \( N \) if \( x \geq 0 \) and \( x = x_1 + x_2 (x_1 \in M, x_2 \in N) \) imply \( x_1 \geq 0 \) and \( x_2 \geq 0 \). A linear subspace \( I \) of a vector lattice \( E \) is called a
lattice ideal if \( y \leq I \) whenever \( x \in I \) and \( |y| \leq |x| \). A lattice ideal \( I \) of \( E \) is a band if \( I \) contains the supremum of every subset of \( I \) that is bounded above in \( E \). Let \( E \) and \( F \) be vector lattices. A linear mapping \( \varphi : E \to F \) is order bounded (order continuous, respectively) if it maps every order bounded set (order convergent net) in \( E \) to an order bounded set (order convergent net) in \( F \). The set of all order bounded (order continuous, respectively) linear mappings from \( E \) into \( F \) is denoted by \( \mathcal{L}(E, F) \) (\( \mathcal{L}_c(E, F) \), respectively) will be denoted by \( E^b \) (\( E^c \), respectively). A linear mapping \( \varphi : E \to F \) is positive if the image of every positive vector under \( \varphi \) is positive. This induces a partial order \( \geq \) in \( \mathcal{L}(E, F) \), which is defined by \( \varphi \geq \psi \) if \( \varphi - \psi \) is positive. F. Riesz verified the following facts: (cf. \([1]\)) If \( E \) is a vector lattice and \( F \) is an order complete vector lattice, then \( \mathcal{L}(E, F) \) forms an order complete vector lattice and \( \mathcal{L}_c(E, F) \) is a closed ideal of \( \mathcal{L}(E, F) \). Every closed ideal of an order complete vector lattice is a band. \( I \) is a band in an order complete vector lattice \( E \) if and only if \( E \) is the order direct sum of \( I \) and another band \( I' \).

2. Extension of order bounded linear mappings.

When \( E \) and \( F \) are complete vector lattices and \( I \) is a closed ideal in \( E \), every order bounded linear mapping: \( I \to F \) can be extended to the whole space \( E \) preserving the order boundedness.

**Lemma.** The projection of every order bounded set of \( E \) into \( I \) is order bounded in \( I \) and the projection of every order convergent sequence into \( I \) is order convergent in \( I \).

**Proof.** Every assertion follows from the fact that a closed ideal is a band and hence \( E \) is represented as an order direct sum \( E = I \oplus I' \).

**Theorem 1.** The natural restriction map \( \theta : \mathcal{L}(E, F) \to \mathcal{L}(I, F) \) defined by \( \theta(\varphi) = \varphi|_I \) (\( \varphi \in \mathcal{L}(E, F) \)) is a surjective linear mapping and \( \theta(\mathcal{L}(E, F)^+) = \mathcal{L}(I, F)^+ \).

**Proof.** For the first assertion of the theorem, it suffices to show that any \( \varphi \in \mathcal{L}(I, F)^+ \) can be extended to an element of \( \mathcal{L}(E, F)^+ \), due to the fact that both \( \mathcal{L}(E, F) \) and \( \mathcal{L}(I, F) \) are complete vector lattices and hence they are decomposed into positive and negative parts. Since \( I \) is a band, \( E \) is represented as an order direct sum \( E = I \oplus I' \). Define \( \bar{\varphi} : E \to F \) by \( \bar{\varphi}(x) = \varphi(x_I) \), where \( x \in E \).
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\[ x_1 + x_2, \quad (x_1 \in I \text{ and } x_2 \in I'). \]

Clearly \( \phi \) maps every positive vector to a positive vector; hence \( \phi \in \mathcal{L}(E, F)^+ \). This gives the required extension. \( \theta(\mathcal{L}(E, F)^+) = \mathcal{L}(I, F)^+ \) is obvious since \( I^+ \subseteq E^+ \).

3. Dual mappings.

For any \( \varphi \in \mathcal{L}(E, F) \), define \( \varphi^* : F^b \to E^b \) by \( \varphi^*(f) = f \circ \varphi \ (\forall f \in F^b) \). Clearly \( \varphi^*(f) \in E^b \), and hence \( \varphi^* \) is well defined. Furthermore, \( \varphi^*(f) \in E^\epsilon \) if \( f \in F^\epsilon \) and \( \varphi \in \mathcal{L}_\epsilon(E, F) \).

**Lemma.** If \( \varphi \in \mathcal{L}(E, F) \), then \( \varphi^* \) is \( \sigma(F^b, F) - \sigma(E^b, E) \) continuous. (cf. [2])

**Theorem 2.** If \( [a, b] \) is an order interval in \( F^b \), then \( \varphi^*[a, b] \) is weak*-closed in \( E^b \).

**Proof.** We shall first show that \( [a, b] \) is weak*-compact in \( F^b \). We note that \( \sigma(F^b, F) \) is the topology on \( F^b \) induced by the product topology on \( R^F = \prod R_x \), where each \( R_x \) is the real line. Therefore, it suffices to show that \( [a, b] \) is compact in \( R^F \). Let a net \( \{a_i\} \) converges to \( a \) in \( R^F \), where each \( a_i \in [a, b] \); or equivalently, \( a_i(x) \to a(x) \) for all \( x \in F \). Then clearly \( a \) is linear. Moreover, for any \( x \in F^+ \) we have \( a(x) \leq a_i(x) \leq b(x) \), which implies \( a(x) \leq a(x) \leq b(x) \). Hence \( a \) is an order bounded linear functional and \( a \in [a, b] \). Therefore, \( [a, b] \) is closed in \( R^F \). We shall reach the conclusion by showing that \( [a, b] \) is a subset of a compact subset of \( R^F \). Denote the set \( \{f \in R^F : \forall x \in F, \ a \in [a, b] : a(x) = f(x) \} \) by \( [a, b] \). Since \( [a, b] \) is a subset of \( [a, b] \) \( (x^+ - x^-) \), which is bounded in \( R^1 \), \( [a, b] \) is compact in \( [a, b] \) is compact by Tychonoff and it contains \( [a, b] \). Hence \( [a, b] \) is compact in \( R^F \). By lemma \( \varphi^*[a, b] \) is weak*-compact in \( E^b \), and hence weak*-closed since \( \sigma(E^b, E) \) is Hausdorff. This completes the proof.

**Corollary.** If \( [a, b] \) is an order interval in \( E^b \), then \( \varphi^*[a, b] \) is weak*-compact in \( E^b \).

**Proof.** An order interval of \( E^b \) is identical with that of \( F^b \). For, consider the order interval \( [a, b] \) in \( F^b \), where \( a, b \in F^c \). If \( z \in [a, b] \), then \( \theta \leq b - z \leq (b - z) + (z - a) = b - a \in F^c \). Since \( F^c \) is a band in \( E^b \), \( b - z \in F^c \) and hence \( z \in F^c \). Therefore, \( [a, b] \subseteq F^c \). By theorem 2 \( \varphi^*[a, b] \) is weak*-compact in \( E^b \).
But since \( \varphi[a,b] \subseteq E^* \) and \( \sigma(E^*, E) \) is the subspace topology induced by \( \sigma(E^*, E) \), \( \varphi[a,b] \) is weak*-compact in \( E^* \), and hence weak* closed. It is not difficult to show that all the theory discussed in this work can be applied also in the \( \mathcal{L}_n(E, F) \).

References


