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ON SOME H-STAR(*) CURVATURE TENSORS IN KAHLER MANIFOLD

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In this paper we shall define some H-curvature tensors in a Kahler manifold M_n which are analogous to the H-conharmonic curvature tensor and H-conformal curvature tensor.

1. Introduction.

Let us consider an n-dimensional Kahler manifold M_n which admits a tensor field F of type (1.1), a Riemannian metric g with Riemannian connexion D satisfying

(1.1) $$\overline{\overline{X}}+X=0,$$

(1.2) $$g(\overline{X}, \overline{Y})=g(X, Y),$$

and

(1.3) $$(D_X F)(Y)=0,$$

where $F(X)=\overline{X}$ and $X, Y, Z,$ are arbitrary vectors.

Let there be a tensor $F(X, Y)$ of type (0, 2) defined by

(1.4) $$F(X, Y)=g(\overline{X}, Y)$$

which satisfies

$$F(X, \overline{Y})=g(X, Y),\ F(X, Y)=-F(Y, X)$$

Let $K(X, Y, Z)$ be the curvature tensor of type (1, 3) then we have

(1.5a) $$K(X, Y, Z)=-K(Y, X, Z),$$

(1.5b) $$K(X, Y, Z)+K(Y, Z, X)+K(Z, X, Y)=0,$$

and

(1.5c) $$(\text{div } K)(X,Y,Z)=(D_X\text{Ric})(Y,Z)-(D_Y\text{Ric})(X,Z).$$

Let

(1.6) $$K(X, Y, Z, T)=g(K(X, Y, Z), T)$$

be the curvature tensor of type (0,4) of M_n.

Let $R(X, Y)$ be the Ricci tensor of type (0, 2) of M_n then

(1.7a) $$R(\overline{X}, \overline{Y})=R(X, Y)$$

(1.7b) $$R(X, Y)=R(Y, X)$$

and

(1.7c) $$X.K=2(\text{div } r)(X) : g(r(X), Y)=R(X, Y),$$

where $r(X)$ and K are Ricci tensor of type (1, 1) and scalar curvature respectively.

The manifold M_n is called recurrent manifold if

(1.8) $$(D_U K)(X, Y, Z)=A(U)K(X, Y, Z),$$

Ricci recurrent manifold if

(1.9) $$(D_U R)(Y, Z)=A(U)R(Y, Z)$$

for the recurrence parameter $A(U)$.

Manifold is called symmetric manifold if

(1.10) $$(D_U K)(X, Y, Z)=0.$$

2. *H*-conharmonic* curvature tensor.

Let us define a tensor $L(X, Y, Z)$ of type (1,3) by the relation

(2.1) $$L(X, Y, Z)=K(X, Y, Z)-\frac{1}{n+4}[g(Y, Z)r(X)-g(X, Z)r(Y) + g(\overline{Y}, Z)r(\overline{X})-g(\overline{X}, Z)r(\overline{Y})-2g(\overline{X}, Y)r(\overline{Z})],$$

and we shall call it as H-conharmonic* curvature tensor.

The manifold M_n will be called H-conharmonic* recurrent if

(2.2) $$(D_U L)(X, Y, Z)=A(U)L(X, Y, Z),$$

for the recurrence parameter $A(U)$ and it will be called H-conharmonic* symmetric if

(2.3) $$(D_U L)(X, Y, Z)=0.$$

Now, we shall study the properties of H-conharmonic* curvature tensor.

By straightforward calculations from (2.1), we have the following identities:

(2.4a) $$L(X, Y, Z)=-L(Y, X, Z),$$

(2.4b) $$(C_1^1 L)(X, Y, Z) \overset{\text{def}}{=} L(Y, Z)=\frac{(n+2)R(Y, Z)-Kg(Y, Z)}{n+4}$$

(2.4c) $$(C_3^1 L)(X, Y, Z)=0,$$

and

(2.4d) $$L(X, Y, Z)+L(Y, Z, X)+L(Z, X, Y)=0,$$

where C_1^1 denotes the contraction in first slot and C_3^1 denotes contraction in third slot.

THEOREM 2.1. *In H-conharmonic* symmetric manifold* M_n, *the scalar curvature is constant.*

PROOF. Taking covariant derivative of (2.1) and using (2.3), we have

(2.5) $$(DK)(X, Y, Z)=\frac{1}{(n+4)}[(D_U r)(X)g(Y, Z)-(D_U r)(Y)g(X, Z)$$

$$+(D_U r)(\overline{X})g(\overline{Y}, Z)-(D_U r)(\overline{Y})g(\overline{X}, Z)-2(D_U r)(\overline{Z})g(\overline{X}, Y)].$$

Transvecting (2.5) by -1_g, the conjugate tensor of g, we have the result using (1.7c).

THEOREM 2.2. *A necessary and sufficient condition for the manifold M_n to be H-conharmonic* symmetric is that it is sysmmetric.*

PROOF. Transvecting (2.5) by -1_g we have

(2.6) $$(D_U r)(X)=0$$

Putting (2.6) in (2.5) we have the necessary part.

Conversely, if it is symmetric then from (1.10), we have

(2.7) $$(D_U r)(X)=0$$

Putting (1.10) and (2.7) in the covariant differential of (2.1), we have the sufficiency part.

THEOREM 2.3. *The manifold M_n of constant holomorphic sectional curvature is H-conharmonically* flat.*

PROOF. The constant holomorphic sectional curvature k of the Kahler manifold M_n is given by

(2,8) $$K(X,Y,Z)=\frac{k}{4}[g(X,Z)Y-g(Y,Z)X+g(\overline{X},Z)\overline{Y}-g(\overline{Y},Z)\overline{X}+2g(\overline{X},Y)\overline{Z}].$$

Since every Kahler manifold of constant holomorphic sectional curvature is an Einstein manifold, therefore

(2.9) $$r(X)=\frac{K}{n}X$$

Putting $k=\frac{4K}{n(n+4)}$ in (2.8) using (2.1) and (2.9), we have $L(X,Y,Z)=0$.

THEOREM 2.4. *A H-conharmonic* recurrent manifold M_n is recurrent if*

(2.10) $$U.K=A(U)K$$

where $A(U)$ is recurrence parameter.

PROOF. Taking covariant derivative of (2.1) and using (1.8) and (2.2), we have

$$A(U)[L(X,Y,Z)-K(X,Y,Z)]=-\frac{1}{n+4}[(D_U r)(X)g(Y,Z)$$
$$-(D_U r)(Y)g(X,Z)+(D_U r)(\overline{X})g(\overline{Y},Z)$$

$$-(D_Ur)(\bar{Y})g(\bar{X},Z)-2(D_Ur)(\bar{Z})g(\bar{X},Y)].$$

Putting $L(X,Y,Z)-K(X,Y,Z)$ from (2.1) and transvecting the result by $-1g$, we have

$$(D_Ur)(X)=A(U)r(X),$$

which gives (2.10).

THEOREM 2.5. *In the manifold M_n if any two of the following properties hold then third also hold*

(i) *It is a recurrent manifold,*

(ii) *it is a H-conharmonic* recurrent,*

(iii) *it is Ricci recurrent,*

for the same recurrence parameters.

PROOF. Taking covariant derivative of (2.1), we have

$$(2.11)\qquad (D_UL)(X,Y,Z)=(D_UK)(X,Y,Z)-\frac{1}{(n+4)}[(D_Ur)(X)g(Y,Z)$$
$$-(D_Ur)(Y)g(X,Z)+(D_Ur)(\bar{X})g(\bar{Y},Z)$$
$$-(D_Ur)(\bar{Y})g(\bar{X},Z)-2(D_Ur)(\bar{Z})g(\bar{X},Z)].$$

With the help of eguations (2.1) and (2.11) it can be shown that if any two of the equations (1.8), (1.9) and (2.2) hold then third one also holds.

THEOREM 2.6. *In the manifold M_n if any two of the following hold then third also holds*:

(i) *It is H-conharmonic recurrent,*

(ii) *it is H-conharmonic* recurrent,*

(iii) *it is Ricci recurrent,*

for the same recurrence parameter.

PROOF. From H-conharmonic curvature tensor (Sinha, 1972), we have

$$(2.12)\qquad S(X,Y,Z)=L(X,Y,Z)+\frac{1}{n+4}[R(X,Z)Y-R(Y,Z)X$$
$$+R(\bar{X},Z)\bar{Y}-R(\bar{Y},Z)\bar{X}+2R(\bar{X},Y)\bar{Z}].$$

From this we have

$$(2.13)\qquad (D_US)(X,Y,Z)=(D_UL)(X,Y,Z)+\frac{1}{n+4}[(D_UR)(X,Z)Y$$
$$-(D_UR)(Y,Z)X+(D_UR)(\bar{X},Z)\bar{Y}$$

$$-(D_U R)(\bar{Y}, Z)\bar{X} + 2(D_U R)(\bar{X}, Y)\bar{Z}].$$

If M_n is H-conharmonic recurrent, then

$$(D_U S)(X, Y, Z) = A(U) S(X, Y, Z), \tag{2.14}$$

where $A(U)$ is recurrence parameter.

With the help of (2.12) and (2.13) it can be shown that if any two of the equations (1.9), (2.2) and (2.14) hold then third also holds.

3. H-conformal* curvature tensor.

Let us define a tensor $M(X, Y, Z)$ of type (1.3) by the relation

$$\begin{aligned} M(X, Y, Z) = K(X, Y, Z) - \frac{1}{n+4} [&r(X)g(Y, Z) - r(Y)g(X, Z) \\ &+ r(\bar{X})g(\bar{Y}, Z) - r(\bar{Y})g(\bar{X}, Z) - 2r(\bar{Z})g(\bar{X}, Y)] \\ &+ \frac{K}{(n+2)(n+4)} [g(Y, Z)X - g(X, Z)Y + g(\bar{Y}, Z)\bar{X} \\ &- g(\bar{X}, Z)\bar{Y} - 2g(\bar{X}, Y)\bar{Z})]. \end{aligned} \tag{3.1}$$

and we shall call it as H-conformal* curvature tensor.

The manifold M_n is said to be H-conformal* recurrent if

$$(D_U M)(X, Y, Z) = A(U) M(X, Y, Z), \tag{3.2}$$

for the recurrence parameter $A(U)$ and it is said to be H-conformal* symmetric if

$$(D_U M)(X, Y, Z) = 0. \tag{3.3}$$

From (3.1), we have the following identities:

(3.4) (1) $M(X, Y, Z) = -M(Y, X, Z)$

(2) $(C_1^1 M)(X, Y, Z) \overset{\text{def}}{=} M(Y, Z) = \frac{n+2}{n+4} R(Y, Z)$

(3) $(C_3^1 M)(X, Y, Z) = 0$

and

(4) $M(X, Y, Z) + M(Y, Z, X) + M(Z, X, Y) = 0$

Following theorems can be proved on the lines of the proof of the theorems in Section 2.

THEOREM 3.1. *In H-conformal* symmetric manifold M_n, the scalar curvature is constant.*

THEOREM 3.2. *A necessary and sufficient condition for the manifold M_n to be H-conformal* symmetric is that it is symmetric.*

THEOREM 3.3. *The manifold M_n of constant holomorphic sectional curvature is H-conformally* flat.*

THEOREM 3.4. *The H-conformal* recurrent manifold M_n is recurrent if*

$$U.K=A(U)K$$

for the recurrence parameter $A(U)$.

THEOREM 3.5. *In the manifold M_n if any two of the following properties hold then third also holds:*

(i) *it is a recurrent manifold,*

(ii) *it is H-conformal* recurrent,*

(iii) *it is Ricci recurrent,*

for the same recurrence parameter.

THEOREM 3.6. *In the manifold M_n if any two of the following hold then third also holds:*

(i) *it is H-conformal recurrent,*

(ii) *it is H-conformal* recurrent,*

(iii) *it is Ricci recurrent,*

for the same recurrence parameter.

4. *H*-conformal** curvature tensor.

Let us define a tensor $N(X,Y,Z)$ of type (1,3) by the relation

$$(4.1)\qquad N(X,Y,Z)=K(X,Y,Z)-\frac{1}{n+4}[R(Y,Z)X-R(X,Z)Y+R(\bar{Y},Z)\bar{X} -R(\bar{X},Z)\bar{Y}-2R(\bar{X},Y)\bar{Z}]+\frac{k}{(n+2)(n+4)}[g(Y,Z)X -g(X,Z)Y+g(\bar{Y},Z)\bar{X}-g(\bar{X},Z)\bar{Y}-2g(\bar{X},Y)\bar{Z}].$$

and we shall call it as H-conformal** curvature tensor.

The manifold M_n is said to be H-conformal** recurrent if

$$(4.2)\qquad (D_UN)(X,Y,Z)=A(U)N(X,Y,Z),$$

where $A(U)$ is recurrence parameter and it is said to be H-conformal** symmetric if

$$(4.3)\qquad (D_UN)(X,Y,Z)=0.$$

Proceeding in the similar manner we have the following theorems on H-conformal** curvature tensor $N(X,Y,Z)$:

THEOREM 4.1. *From equation* (4.1), *we have*

(i) $N(X,Y,Z)=-N(Y,X,Z)$

(ii) $(C_1^1N)(X,Y,Z) \overset{\text{def}}{=} N(Y,Z)=\frac{2R(Y,Z)+Kg(Y,Z)}{n+4}$

(iii) $(C_3^1N)(X,Y,Z)=0,$

(iv) $N(X,Y,Z)+(Y,Z,X)+N(Z,X,Y)=0.$

REMARK 4.1. From (ii) (theorem 4.1), we see that for an Einstein manifold

$$M(Y,Z)=N(Y,Z).$$

THEOREM 4.2. *In H-conformal** symmetric manifold M_n, the scalar curvature is constant.*

THEOREM 4.3. *A necessary and sufficient condition for the manifold M_n to be H-conformal** symmetric is that it is symmetric.*

THEOREM 4.4. *The manifold M_n of constant holomorphic sectional curvature is H-conformally** flat.*

THEOREM 4.5. *A H-conformal** recurrent manifold M_n is recurrent if*

$$U.K=A(U)K,$$

with $A(U)$ as recurrence parameter.

THEOREM 4.6. *In the manifold M_n if any two of the following properties hold, then third also holds:*

(i) *it is a recurrent manifold,*

(ii) *it is H-conformal** recurrent,*

(iii) *it is Ricci recurrent,*

for the same recurrence paramater.

THEOREM 4.7. *In the manifold M_n if any two of the following hold then third also holds:*

(i) *it is H-conformal recurrent,*

(ii) *it is H-conformal** recurrent,*

(iii) *it is Ricci recurrent,*

for the same recurrence parameter.

REMARK 4.2. Proof of the theorems (4.2) to (4.7) are similar to that of the theorems (3.1) to (3.6).

THEOREM 4.8. *In the manifold M_n the H-conformal curvature, H-conharmonic* *curvature, H-conformal* curvature, H-conformal** curvature and curvature tensors are linearly dependent.*

PROOF. H-conformal curvature tensor (Tachibana, 1967) is given by

$$C(X,Y,Z)=K(X,Y,Z)-\frac{1}{n+4}[R(Y,Z)X-R(X,Z)Y+R(\overline{Y},Z)\overline{X} -R(\overline{X},Z)\overline{Y}+r(\overline{X})g(\overline{Y},Z)-r(Y)g(X,Z) +r(X)g(Y,Z)-r((\overline{Y})g(\overline{X},Z)-2R(\overline{X},Y)\overline{Z} -2r(\overline{Z})g(\overline{X},Y)]+\frac{K}{(n+2)(n+4)}[g(Y,Z)X -g(X,Z)Y+g(\overline{Y},Z)\overline{X}-g(\overline{X},Z)\overline{Y}-2g(\overline{X},Y)\overline{Z}], \tag{4.4}$$

and H-conharmonic curvature tensor (Sinha, 1972) is given by

$$S(X,Y,Z)=K(X,Y,Z)-\frac{1}{n+4}[R(Y,Z)X-R(X,Z)Y +R(\overline{Y},Z)\overline{X}-R(\overline{X},Z)\overline{Y}+r(X)g(Y,Z)-r(Y)g(X,Z) +r(\overline{X})g(\overline{Y},Z)-r(\overline{Y})g(\overline{X},Z)-2R(\overline{X},Y)\overline{Z} -2r(\overline{Z})g(\overline{X},Y)]. \tag{4.5}$$

From (4.4), (4.5), (4.1) and (3.1), we have

$$2C(X,Y,Z)=S(X,Y,Z)+M(X,Y,Z)+N(X,Y,Z)-K(X,Y,Z) \tag{4.6}$$

which proves the statement.

THEOREM 4.9. *In H-conformal* (or H-conformal**) recurrent manifold M_n if it is H-conformal recurrent, and it is H-conharmonic recurrent, then it is H-conformal** (or H-conformal*) recurrent, for the same recurrence parameter.*

PROOF. It can be proved using theorems (4.6), (4.7) and equation (4.6).

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