ON SOME FIXED POINT THEOREMS

By R.N. Mukherjee

1.1 Introduction.

In [1] Ray has given a theorem on fixed point of mappings in a metric space. Theorem is the following.

THEOREM A. Let $T_1$ and $T_2$ be maps, each mapping a complete metric space $(E, d)$ into itself such that

(i) $d(T_1 x, T_2 y) \leq \alpha d(x, y)$; where $0 < \alpha < 1$ and $x, y$ belong to $E$ ($x \neq y$) and

(ii) there is a point $x_0$ in $E$ such that any two consecutive members of $\{x_1 = T_1 x_0, x_2 = T_2 x_1, x_3 = T_1 x_2, x_4 = T_2 x_3, \ldots\}$ are distinct, then $T_1$ and $T_2$ have a unique common fixed point in $E$.

We give below the definition of $\varepsilon$-chainable metric space as in reference [2].

DEFINITION. A metric space $(E, d)$ will be said to be $\varepsilon$-chainable if for every $x, y$ belonging to $E$ there exists an $\varepsilon$-chain, i.e. a finite set of points $x = x_0, x_1, x_2, \ldots, x_n = y$ ($n$ may depend both on $x$ and $y$), such that $d(x_{i-1}, x_i) < \varepsilon$, ($i = 1, 2, \ldots, n$).

We prove next the following theorem.

THEOREM 1.1. Let $E$ be a complete $\varepsilon$-chainable metric space; and let $T_1$ and $T_2$ be two maps each mapping $E$ into itself such that if $0 < d(x, y) \leq \varepsilon$, then,

(i) $d(T_i x, T_i y) \leq \alpha d(x, y); i = 1, 2.$

(ii) $d(T_i x, T_j y) \leq \alpha d(x, y); i \neq j.$

where in (i) and (ii) $x, y$ belong to $E$ ($x \neq y$) and $0 < \alpha < 1$. Also $T_1$ and $T_2$ satisfy the condition (ii) of theorem A; then $T_1$ and $T_2$ have a common fixed point in $E$.

PROOF. Since $(E, d)$ is $\varepsilon$-chainable we define for $x, y$ belonging to $E$

$$d_{\varepsilon}(x, y) = \inf \sum_{i=1}^{n} d(x_i, x_{i-1}),$$

where the infimum is taken over all $\varepsilon$-chains $x_0, x_1, x_2, \ldots, x_n$ joining $x_0 = x$ and $x_n = y$. Then $d$ is a distance function in $E$ satisfying:
(1) \( d(x, y) < d_\varepsilon(x, y) \)

(2) \( d(x, y) = d_\varepsilon(x, y) \) for \( d(x, y) < \varepsilon \).

From (2) it follows that a sequence \( \{x_n\} \), \( x_n \in E \) is a Cauchy sequence with respect to \( d_\varepsilon \) if and only if it is a Cauchy sequence with respect to \( d \) and is convergent with respect to \( d_\varepsilon \) if and only if it is convergent with respect to \( d \). Since \( (E, d) \) is complete therefore \( (E, d_\varepsilon) \) is a complete metric space. Moreover the following is true.

Given \( x, y \) in \( E \) and any \( \varepsilon \)-chain \( x_0, x_1, x_2, \ldots, x_n \) with \( x_0 = x \) and \( x_n = y \) such that \( d(x_i, x_{i-1}) < \varepsilon \) \( (i = 1, 2, \ldots, n) \) we have (if \( n \) is even, say \( n = 2 \))

\[
d(T_1 x_0, T_1 x_1) \leq \alpha d(x_0, x_1) < \varepsilon
\]

\[
d(T_2 x_2, T_1 x_1) \leq \alpha d(x_2, x_1) < \varepsilon
\]

so that \( T_1 x_0, T_1 x_1, T_2 x_2 \) form an \( \varepsilon \)-chain for \( T_1 x_0 \) and \( T_2 x_2 \). Similarly if \( n \) is odd (say \( n = 3 \)) we can show that \( T_1 x_0, T_2 x_1, T_1 x_2, T_2 x_3 \) form an \( \varepsilon \)-chain for \( T_1 x_0, T_2 x_3 \).

Condition (i) is also necessary which can be seen by taking \( n = 4 \).

Combining all the cases above it can be shown that

\[
d_\varepsilon(T_1 x, T_2 y) \leq \alpha \sum_{i=1}^{n} d(x_{i-1}, x_i)
\]

\( x_0, x_1, x_2, \ldots, x_n = y \) being an arbitrary \( \varepsilon \)-chain,

therefore we have

\[
d_\varepsilon(T_1 x, T_2 y) \leq \alpha d_\varepsilon (x, y),
\]

and since \( T_1 \) and \( T_2 \) also satisfy the condition (ii) of theorem A therefore by the conclusion of theorem A we have the required result.

**THEOREM 1.2.** Let \( E \) be a complete \( \varepsilon \)-chainable metric space; and let \( T_1 \) and \( T_2 \) be two maps each mapping \( E \) into itself. If there exists two integers \( p_1 \) and \( p_2 \), such that if \( 0 < d(x, y) < \varepsilon \), then

(i) \( d(T_1^{p_1} x, T_1^{p_2} y) \leq \alpha d(x, y) ; i = 1, 2 ; \)

(ii) \( d(T_1^{p_i} x, T_2^{p_j} y) \leq \alpha d(x, y) ; i \neq j ; \)

where in (i) and (ii) \( x, y \in E \) \((x \neq y) \) and \( 0 < \alpha < 1 \). Also let \( T_1^{p_1} \) and \( T_2^{p_2} \) satisfy the condition (ii) of theorem A, then \( T_1 \) and \( T_2 \) have a common fixed point.

**PROOF.** Set \( S_1 = T_1^{p_1} \) and \( S_2 = T_2^{p_2} \). Then by theorem 1.1 there exists a unique fixed point \( x \) such that \( S_1(x) = S_2(x) = x \), i.e., \( T_1^{p_1}(x) = x = T_2^{p_2}(x) \). From which it follows that \( T_1^{p_1+1}(x) = T_1(x) \), which implies that \( T_1^{p_1}(T_1(x)) = T_1(x) \), since \( T_1^{p_1} \)
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has a unique fixed point, therefore \( T_1(x) = x \). Similarly it follows that \( T_2(x) = x \).

**THEOREM 1.3.** Let \( E \) be a complete \( \varepsilon \)-chainable metric space. Let \( T_1 \) and \( T_2 \) be two mappings of \( E \) into itself and suppose there exists a mapping \( K \) of \( E \) into itself such that \( K \) has a right inverse \( K^{-1} \) (i.e., a function \( K \) such that \( KK^{-1} = I \), where \( I \) is the identity mapping of \( E \)) and if \( 0 < d(x, y) < \varepsilon \), then

(i) \( d(K^{-1}T_iKx, K^{-1}T_iKy) \leq \alpha d(x, y) \); \( i = 1, 2 \);

(ii) \( d(K^{-1}T_iKx, K^{-1}T_jKy) \leq \alpha d(x, y) \); \( i \neq j \);

where in (i) and (ii) \( x, y \in E (x \neq y) \) and \( 0 < \alpha < 1 \). Also suppose \( K^{-1}T_1K \) and \( K^{-1}T_2 \) satisfy the condition (ii) of theorem A. Then \( T_1 \) and \( T_2 \) possess a common fixed point which is unique.

**PROOF.** Set \( K^{-1}T_1K = S_1 \) and \( K^{-1}T_2K = S_2 \), then \( S_1 \) and \( S_2 \) have a common fixed point which is unique, by theorem 1.1. i.e., \( K^{-1}T_1K(x) = x = K^{-1}T_2K(x) \) from which we get \( KK^{-1}T_1K(x) = K(x) \), therefore \( T_1(K(x)) = K(x) \). Similarly \( T_2(K(x)) = K(x) \), in other words \( T_1 \) and \( T_2 \) have a common fixed point \( K(x) \).

Next we give a theorem on sequence of mappings and their fixed points.

**THEOREM 1.4.** Let \((E, d)\) be a complete metric space and let \( T_{1k}^1 \) and \( T_{2k}^2 \) be two sequences of mappings each mapping \( E \) into itself such that

\[
d(T_{1k}^1x, T_{2k}^2y) \leq \beta_k [d(T_{1k}^1x, x) + d(T_{2k}^2y, y)]
\]

where all \( \beta_k \)'s are \( < \frac{1}{2} \) and positive, and \( x, y \in E \) \((x \neq y) \) \((k = 1, 2, \ldots) \). Let \( T^1 \) and \( T^2 \) be mappings such that \( \lim_{k \to \infty} d(T_{1k}^1x, T^1x) = 0 \) and \( \lim_{k \to \infty} d(T_{2k}^2x, T^2x) = 0 \), for all \( x \) in \( E \). Also \( \beta_k \to \beta \) \( (0 < \beta < \frac{1}{2}) \) as \( k \to \infty \). Then \( T^1 \) and \( T^2 \) have a common fixed point (say \( u \)). If \( u_k \) for a fixed \( k \) is the simultaneous fixed point of \( T_{1k}^1 \) and \( T_{2k}^2 \) (which exists because of a theorem in [4]) then \( \lim_{k \to \infty} u_k = u \).

**PROOF.** To prove the first part of the theorem we have only to show that \( T^1 \) and \( T^2 \) satisfy the inequality (*) as given below.

Now,

\[
d(T^1x, T^2y) \leq d(T^1x, T_{1k}^1x) + d(T_{1k}^1x, T_{2k}^2y) + d(T_{2k}^2y, T^2y)
\]
\[
\begin{align*}
\leq & \; d(T^1 x, T^1 y) + d(T^2 y, T^2 y) + \beta_k \{ d(T^1 x, x) + d(T^2 y, y) \} \\
\leq & \; d(T^1 x, T^1 x) + d(T^2 y, T^2 y) + \beta_k \{ d(T^1 x, T^1 x) \\
& + d(T^1 x, x) + d(T^2 y, T^2 y) + d(T^2 y, y) \}
\end{align*}
\]

therefore as \( k \to \infty \), since

\[
d(T^1 x, T^1 y) \to 0 \quad \text{and also} \quad d(T^2 x, T^2 y) \to 0
\]

we have

\[
d(T^1 x, T^2 y) \leq \beta \{ d(T^1 x, x) + d(T^2 y, y) \} \quad (\ast)
\]

Then by the theorem given in [3] \( T^1 \) and \( T^2 \) both have a simultaneous fixed point in \( E \) (which we denote by \( u \) say). Now to prove that \( \lim_{k \to \infty} u_k = u \) we proceed as follows: Since \( u \) is in \( E \), fixing \( n=n_0 \) we form the following sequence

\[
x_1 = T^1_{n_0} (u), \quad x_2 = T^2_{n_1} (x_1), \quad x_3 = T^1_{n_1} (x_2), \quad x_4 = T^2_{n_2} (x_3), \ldots
\]

then after little calculation as done in [4], it can be shown that

\[
d(x_k, x_{k+1}) \leq \left( \frac{\beta_{n_k}}{1-\beta_{n_k}} \right)^k d(u, T^1_{n_k} (u))
\]

from this it follows that

\[
d(x_k, x_{k+p}) \leq \frac{r^k}{1-r} d(u, T^1_{n_k} (u))
\]

where \( r = \frac{\beta_{n_k}}{1-\beta_{n_k}} \).

therefore there exists \( u_{n_k} \) such that \( \lim_{k \to \infty} u_k = u_{n_k} \). Now to show that

\[
T^1_{n_k} (u_{n_k}) = u_{n_k} = T^2_{n_k} (u_{n_k}).
\]

We need the following inequality

\[
d(u_{n_k}, T^1_{n_k} (u_{n_k})) \leq d(u_{n_k}, x_k) + d(x_k, T^1_{n_k} (u_{n_k}))
\]

\[
= d(u_{n_k}, x_k) + d(T^2_{n_k} (x_{k-1}), T^1_{n_k} (u_{n_k})),
\]

where we choose \( k \) to be even positive integer.

Therefore

\[
d(u_{n_k}, T^1_{n_k} (u_{n_k})) \leq d(u_{n_k}, x_k) + \beta_{n_k} \{ d(x_{k-1}, T^2_{n_k} (x_{k-1})) + d(u_{n_k}, T^1_{n_k} (u_{n_k})) \}
\]

i.e., \( (1-\beta_{n_k}) d(u_{n_k}, T^1_{n_k} (u)) \leq d(u_{n_k}, x_k) + \beta_{n_k} d(x_{k-1}, x_k) \)

and letting \( k \to \infty \), we can prove that \( T^1_{n_k} (u_{n_k}) = u_{n_k} \). That having proved we start with the following inequality,
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\[ d(u_{n_0}, T_{n_0}^2(u_{n_0})) \leq d(u_{n_0}, x_1) + d(x_1, T_{n_0}^2(u_{n_0})) \]
\[ = d(u_{n_0}, T_{n_0}^1(u)) + d(T_{n_0}^1(u), T_{n_0}^2(u_{n_0})) \]
\[ \leq d(u_{n_0}, T_{n_0}^1(u)) + \beta \eta_0 [d(u, T_{n_0}^1(u)) + d(u_{n_0}, T_{n_0}^2(u_{n_0}))] \]

i.e.,
\[ (1-\beta) \cdot d(u_{n_0}, T_{n_0}^2(u_{n_0})) \leq d(u_{n_0}, T_{n_0}^1(u)) + d(T_{n_0}^1(u), T_{n_0}^2(u)) + \beta \eta_0 d(u, T_{n_0}^1(u)) \]

i.e.,
\[ (1-\beta) \cdot d(u_{n_0}, T_{n_0}^2(u_{n_0})) \leq d(u_{n_0}, u) + d(T_{n_0}^1(u), T_{n_0}^2(u)) + \beta \eta_0 d(T(u), T_{n_0}^1(u)) \]

therefore as \( n_0 \to \infty \), \( d(u_{n_0}, u) \to 0 \): which completes the proof.

2.1 In this section we give certain fixed point theorems on a generalized complete metric space. We give below the definition and the characterization of a generalized complete metric space as given in [5]. Theorem B and C mentioned above have also been given in [5].

**DEFINITION.** The pair \((E, d)\) is called a **generalized complete metric space** if \(E\) is a non-void set and \(d\) is a function from \(E \times E\) to extended real numbers satisfying the following conditions:

- \((D0)\) \(d(x, y) \geq 0\)
- \((D1)\) \(d(x, y) = 0\) iff \(x = y\)
- \((D2)\) \(d(x, y) = d(y, x)\)
- \((D3)\) \(d(x, y) \leq d(x, z) + d(z, y)\)
- \((D4)\) every \(d\)-cauchy sequence in \(E\) is \(d\)-convergent, i.e.,

if \(\{x_n\}\) is a sequence in \(E\) such that \(\lim_{m, n \to \infty} d(x_n, x_m) = 0\), then there is an \(x \in E\) with \(\lim_{n \to \infty} d(x_n, x) = 0\). For convenience we will say that a pair \((E, d)\) is a generalized metric space if all but \((D4)\) of the above conditions are satisfied. Let \(\{(E_\alpha, d_\alpha) | \alpha \in O\}\) be a family of disjoint metric spaces. Then there is a natural way of getting a generalized metric space \((E, d)\) from \(\{(E_\alpha, d_\alpha) | \alpha \in O\}\) as follows.

For any \(x, y \in E\) define
\[ d(x, y) = d_\alpha(x, y) \] if \(x, y \in E_\alpha\) for some \(\alpha \in O\)
\[ = +\infty \] if \(x \in E_\alpha\) and \(y \in E_\beta\) for some \(\alpha, \beta \in O\) with \(\alpha \neq \beta\).
Clearly $(E, d)$ is a generalized metric space. Moreover if $(E_{\alpha}, d_{\alpha})$ is complete, then $(E, d)$ is a generalized complete metric space. The main purpose of the above procedure is to show that the above method is the only way to obtain generalized complete metric spaces.

Let $(E, d)$ be a Generalized Complete Metric Space. Define $\sim$ on $E$ as follows, $x \sim y$ iff $d(x, y) < \infty$. Then $\sim$ is an equivalence relation on $E$. Therefore $E$ is decomposed (uniquely) into disjoint equivalence classes $E_{\alpha}$, $\alpha \in O$. From henceforth we will reserve the term 'Canonical decomposition' for the type of decomposition as shown above.

**Theorem B.** Let $(E, d)$ be a generalized metric space. $E = \bigcup \{E_{\alpha} | \alpha \in O\}$ the canonical decomposition and $d_{\alpha} = d | E_{\alpha} \times E_{\alpha}$ for each $\alpha \in O$. Then

(a) for $\alpha \in O$, $(E_{\alpha}, d_{\alpha})$ is a metric space.

(b) for any $\alpha, \beta \in O$ with $\alpha \neq \beta$, $d(x, y) = +\infty$ for any $x \in E_{\alpha}$ and $y \in E_{\beta}$.

(c) $(E, d)$ is a generalized complete metric space iff for each $\alpha \in O$, $(E_{\alpha}, d_{\alpha})$ is a complete metric space.

**Theorem C.** Let $(E, d)$ be a generalized metric space. $E = \bigcup \{E_{\alpha} | \alpha \in O\}$ the canonical decomposition and let $T : E \to E$ be a mapping such that $d(T(x), T(y)) < \infty$ whenever $x, y \in E$. Then $T$ has a fixed point iff $T_{\alpha} = T | E_{\alpha} : E_{\alpha} \to E_{\alpha}$ has a fixed point for some $\alpha \in O$.

We note that (*) is necessary for $T$ to be a mapping from $E_{\alpha} \to E_{\alpha}$.

We prove next the following theorems.

**Theorem 2.1.** Let $(E, d)$ be a generalized complete metric space; $E = \bigcup \{E_{\alpha} | \alpha \in O\}$ be the canonical decomposition. Let $T_{1} : E \to E$ and $T_{2} : E \to E$ be two mappings such that

(a) $d(T_{1}x, T_{2}y) \leq \beta[d(x, T_{1}x) + d(y, T_{2}y)]$ for all $x, y \in E$ and $0 < \beta < \frac{1}{2}$,

(b) $d(T_{i}x, T_{i}y) \leq d(x, y)$ for all $x, y \in E$ ($x \neq y$), $i = 1, 2$

$d(T_{i}x, T_{j}y) \leq d(x, y)$ for all $x, y \in E$ ($x \neq y$), $i \neq j$;

if there exists an $x_{0} \in E$ such that $d(x_{0}, T_{i}(x_{0})) < \infty$ for $i = 1$ or $2$ then for some $\alpha \in O$, the restrictions

$T_{1\alpha} : T_{1}|E_{\alpha} : E_{\alpha} \to E_{\alpha}$ and $T_{2\alpha} : T_{2}|E_{\alpha} : E_{\alpha} \to E_{\alpha}$

satisfy the condition (a) above.
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PROOF. Let $d(x_0, T_1(x_0)) < \infty$, then both $x_0$ and $T_1(x_0) \in E_{\alpha_1}$ for some $\alpha_1 \in \mathbb{O}$. Because of (b) we have if $x_1 \in E_{\alpha_1}$

$$d(T_1x_1, T_1x_0) \leq d(x_1, x_0) < \infty$$

also

$$d(T_2x_1, T_1x_0) \leq d(x_1, x_0) < \infty$$

and therefore $T_1(E_{\alpha_1}) \subseteq E_{\alpha_1}$ and $T_2(E_{\alpha_1}) \subseteq E_{\alpha_1}$. From which it follows that $T_{1\alpha_1}$ and $T_{2\alpha_1}$ satisfy the condition (b) in $E_{\alpha_1}$.

THEOREM 2.2. Assuming the same type of hypothesis as in theorem 2.1 above let $x \in E$ and consider the sequence of successive approximation

$$x_1 = x, \ T_1(x_1) = x_2, \ x_3 = T_2(x_2), \ x_4 = T_1(x_3), \ x_5 = T_2(x_4), \cdots.$$ 

Then the following alternative holds: either

(A) for every $m=0, 1, 2, \cdots$, one has $d(x_m, x_{m+1}) = +\infty$

or (B) the sequence \{x_m\} is $d$-convergent to a simultaneous fixed point of $T_1$ and $T_2$.

PROOF. If (A) does not hold then for some $m$

$$d(x_m, x_{m+1}) < \infty$$

letting $x_m = x_0$, the theorem 2.1 shows that $T_1(E_{\alpha_1}) \subseteq E_{\alpha_1}$ and $T_2(E_{\alpha_1}) \subseteq E_{\alpha_1}$ where $E_{\alpha}$ is the complete metric space containing $x_0$. Therefore by the theorem mentioned in section 1 of the present note also given in [4], the sequence $x_m, x_{m+1}, \cdots$ is $d$-convergent to a simultaneous fixed point of $T_{1\alpha}$ and $T_{2\alpha}$. This implies that (B) holds.

COROLLARY. Assuming the same hypothesis for the case of mappings $T_1 = T_2 = T$ (say), let $x \in E$ and consider the sequence of successive approximation with initial value $x$.

$$x, \ Tx, \ T^2x, \ T^3x, \cdots, \ T^nx, \cdots$$

then following alternative holds, either

(A) for every $m=0, 1, \cdots$, one has $d(T^mx, T^{m+1}x) = +\infty$

or (B) the sequence, $x, \ Tx, \ T^2x, \cdots, \ T^nx, \cdots$ is $d$-convergent to a fixed point of $T$.

PROOF. Proof is in the same line as in theorem 2.2.

Suppose $T$ be a mapping from a metric space $E$ into itself. Also suppose $T$ satisfies the conditions (A0) and (A1) below.
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(A0) \[ d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y) \]
where \(0 < \alpha + \beta + \gamma < 1\).

(A1) \[ d(Tx, Ty) \leq d(x, y), \text{ for all } x, y \in E(x \neq y) \]

then from a theorem of Reich [6] it follows that such a \(T\) has a unique fixed point in \(E\).

**THEOREM 2.3.** Let \(E\) be a generalized complete metric space. Suppose \(T\) be a mapping from \(E\) into itself such that it satisfies conditions (A0) and (A1) above. Let \(x \in E\) and consider the sequence of successive approximation:

\[ x_1 = x, \quad Tx_1 = x_2; \quad T^2x_1 = x_3; \quad T^3x_1 = x_4, \ldots \]

then the following alternative holds, either

(a) for every \(m = 0, 1, 2, \ldots\), one has \(d(x_m, x_{m+1}) = \infty\)

or

(b) the sequence \(\{x_m\}\) is \(d\)-convergent to a fixed point of \(T\).

**PROOF.** Proof is similar to the proof of theorem 2.2 above; only in this case we apply finally the theorem of Reich [6].

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**REFERENCES**


