Kyungpook Math. J.
Volume 14, Number 1
June, 1974

A NOTE ON COUNTABLY PARACOMPACT SPACES

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In the present paper, we shall obtain some characterisations of countably paracompact spaces. First, some characterisations corresponding to the characterisations of paracompact spaces obtained by H. Tamano [9], J.E. Vaughan [10], M.K. Singal and Shashi Prabha Arya [6] will be obtained. Using some of the concepts introduced by J.R. Boone [1], some characterisations will be obtained for certain classes of k-spaces. Analogous to the four weaker forms of paracompactness introduced by Boone, four weaker forms of countable paracompactness will be introduced and it will be shown that in normal spaces all of them become characterisations of countable paracompactness.

1. Characterisations of countably paracompact spaces.

DEFINITION 1.1. Let $\mathscr{U}$ and $\mathscr{V}$ be two families of subsets of a space X. $\mathscr{U}$ is said to be *cushioned* in $\mathscr{V}$ with cushion map $f:\mathscr{U}\to\mathscr{V}$ if for every subfamily $\mathscr{U}'$ of $\mathscr{U}$ we have

$$\overline{\bigcup\{U:U\in\mathscr{U}'\}}\subset\bigcup\{f(U);U\in\mathscr{U}'\}.$$

If '$\leqslant$' is a linear-order on $\mathscr{U}$, then a subfamily $\mathscr{U}'$ of $\mathscr{U}$ is said to be majorized if there exists $U\in\mathscr{U}$ such that $U'\leqslant U$ for every $U'\in\mathscr{U}'$. $\mathscr{U}$ equipped with a linear-order '$\leqslant$' is said to be *linearly cushioned* in $\mathscr{V}$ with cushion map $f:\mathscr{U}\to\mathscr{V}$ if for every majorized subfamily $\mathscr{U}'$ of $\mathscr{U}$ we have

$$\overline{\bigcup\{U:U\in\mathscr{U}'\}}\subset\bigcup\{f(U):U\in\mathscr{U}'\}.$$

The family $\mathscr{U}$ equipped with a well-order '$\leqslant$' is said to be *order cushioned* in $\mathscr{V}$ with cushion map $f:\mathscr{U}\to\mathscr{V}$ if for every $U\in\mathscr{U}$ and every subfamily $\mathscr{U}'$ of $\{V:V<U\}$, we have

$$\mathrm{cl}_u[\bigcup\{V\cap U:V\in\mathscr{U}'\}]\subset\bigcup\{f(V):V\in\mathscr{U}'\}.$$

Here cl_u denotes the closure in the relative topology of U. The family $\mathscr{U}$ equipped with a well-order '$<$' is said to be *well-ordered cushioned* in $\mathscr{V}$ with cushion map $f:\mathscr{U}\to\mathscr{V}$ if for every $U\in\mathscr{U}$ and every subfamily $\mathscr{U}'$ of $\{V:V<U\}$, we have

$$U\cap\overline{\bigcup\{V:V\in\mathscr{U}'\}}\subset\bigcup\{f(V):V\in\mathscr{U}'\}.$$

The concept of cushioned families is due to E-Michael [3] and that of linearly cushioned families is due to H. Tamano [9]. However, the definition of linearly cushioned families as given above is that used by J.E. Vaughan. Tamano's definition differs from that of Vaughan in as much as he required the order to be a well-order instead of a linear-order and he considered bounded subcollections instead of majorized. The concept of order cushioned families is due to J. E. Vaughan [10]. It should be noted that if a family $\mathscr{U}$ is well-ordered cushioned in a family $\mathscr{V}$ then it is also order cushioned in $\mathscr{V}$. However, if $\mathscr{U}$ consists of open sets then $\mathscr{U}$ is order cushioned in $\mathscr{V}$ if and only if it is well-ordered cushioned in $\mathscr{V}$.

THEOREM 1.1 *For a space X, the following are equivalent*:

(a) *X is normal and countably paracompact.*

(b) *Every countable open covering of X has an open cushioned refinement.*

(c) *Every countable open covering of X has a σ-cushioned open refinement.*

(d) *Every countable open covering of X has a linearly cushioned open refinement.*

(e) *Every countable open covering of X has an order cushioned (or well-ordered cushioned) open refinement.*

(f) *Every countable open covering of X has a cushioned refinement.*

PROOF. (a)$\Rightarrow$(b). Let $\mathscr{U}=\{U_i : i\in N\}$ be any countable open covering of X. Since X is countably paracompact, there exists a locally finite, open refinement $\mathscr{V}=\{V_\alpha:\alpha\in\Lambda\}$ of $\mathscr{U}$. since X is normal, $\mathscr{V}$ is shrinkable. Therefore, there exists another locally finite, open covering $\mathscr{W}=\{W_\alpha:\alpha\in\Lambda\}$ of X such that $\overline{W}_\alpha\subset V_\alpha$ for each $\alpha\in\Lambda$. $\mathscr{W}$ is then an open cushioned refinement of $\mathscr{U}$.

(b)$\Rightarrow$(c). Obvious.

(c)$\Rightarrow$(d). Every σ-cushioned refinement is linearly cushioned. Hence the implication.

(d)$\Rightarrow$(e). Let $\mathscr{U}$ be any countable, open covering of X.

Let $\mathscr{V}$ be a linearly cushioned open refinement of $\mathscr{U}$ with respect to a well order '$<$' with cushion map $f:\mathscr{V}\to\mathscr{U}$ (there is no loss of generality in assuming the order to be a well-order in view of J. E. Vaughan [10, theorem 1]. Let $V\in\mathscr{V}$ and let $\mathscr{V}'$ be a subfamily of $\{W:W<V\}$. Since $\mathscr{V}'$ is majorized and $\mathscr{V}$ is linearly cushioned in $\mathscr{U}$, therefore we have

$$\overline{\bigcup\{W:W\in\mathscr{V}'\}}\subset\bigcup\{f(W):W\in\mathscr{V}'\}.$$

Now, $$\mathrm{cl}_V[\bigcup\{V\cap W:W\in\mathscr{V}'\}]=\mathrm{cl}_V[\bigcup(\{W:W\in\mathscr{V}'\})\cap V]=$$

$$\overline{\bigcup\{W:W\in\mathscr{V}'\}\cap V}\cap V\subset\overline{\bigcup\{W:W\in\mathscr{V}'\}}\cap V\subset\overline{\bigcup\{W:W\in\mathscr{V}'\}}$$

$\subset\bigcup\{f(W):W\in\mathscr{V}'\}$. Hence $\mathscr{V}$ is order cushioned in $\mathscr{U}$.

(e)$\Rightarrow$(f). Let $\mathscr{F}$ be any countable, open covering of X. Let $\mathscr{G}$ be an order cushioned open refinement of $\mathscr{F}$ with cushion map $f:\mathscr{G}\to\mathscr{F}$ with respect to the well-order '$<$'. For each $G\in\mathscr{G}$, Let $H_G=G\sim\bigcup\{G':G'<G, G'\in\mathscr{G}\}$. Let $\mathscr{H}=\{H_G$:

$G\in\mathscr{G}\}$. Then $\mathscr{H}$ is a covering of X, for if $x\in X$, then there exists a smallest $G\in\mathscr{G}$ such that $x\in G$ and hence $x\in H_G$. It will be shown that $\mathscr{H}$ is cushioned in $\mathscr{F}$ with cushion map $g:\mathscr{H}\to\mathscr{F}$ such that $g(H_G)=f(G)$. To show this, let $\mathscr{H}'$ be any subfamily of $\mathscr{H}$. Let $y\in\overline{\cup\{H_{G'}:H_{G'}\in\mathscr{H}'\}}$. Since $\mathscr{G}$ is a covering X, there exists $G\in\mathscr{G}$ such that $y\in G$. Now, $G\cap H_{G*}=\phi$ for all $G^*>G$. Therefore if $\mathscr{H}''=\{H_{G'}:G'<G$ and $H_{G'}\in\mathscr{H}'\}$, then $y\in\overline{\cup\{H_{G'}:H_{G'}\in\mathscr{H}''\}}\subset\overline{\cup\{G':G'<G, H_{G'}\in\mathscr{H}''\}}$. Now if $\mathscr{G}'=\{G':H_{G'}\in\mathscr{H}''\}$, then since $\mathscr{G}$ is order cushioned in $\mathscr{F}$ and $G'<G$ for all $G'\in\mathscr{G}'$, therefore we must have $\mathrm{cl}_G[\cup\{G'\cap G:G'\in\mathscr{G}'\}]\subset\cup\{f(G'):G'\in\mathscr{G}'\}$. Now, $\cup\{f(G'):G'\in\mathscr{G}'\}=\cup\{g(H_{G'}):H_{G'}\in\mathscr{H}'\}$.

Also, G being an open set, we have $\mathrm{cl}_G[(\cup\{G':G'\in\mathscr{G}'\})\cap G=\overline{\cup\{G':G'\in\mathscr{G}'\}}\cap G$. This shows that $y\in\mathrm{cl}_G[(\cup\{G':G'\in\mathscr{G}'\}\cap G)]$ and hence $y\in\cup\{g(H_{G'}):H_{G'}\in\mathscr{H}'\}$.

It follows that for every subfamily $\mathscr{H}'$ of $\mathscr{H}$, we have $\overline{\cup\{H_{G'}:H_{G'}\in\mathscr{H}'\}}\subset\cup\{g(H_{G'}):H_{G'}\in\mathscr{H}'\}$. Hence $\mathscr{H}$ is a cushioned refinement of $\mathscr{F}$.

(f)$\Rightarrow$(a). If every countable, open covering of X has a cushioned refinement, then X is normal [3]. Hence X is countably paracompact [7, theorem 2]. This completes the proof of the theorem.

DEFINITION 1.2. A space X is said to have *W-weak topology* with respect to a family $\mathscr{G}$ of subsets of X if a subset U of X is open if and only if $U\cap G$ is open in G for each $G\in\mathscr{G}$. A family $\mathscr{F}$ of subsets of X is said to be an *F-hereditary collection* if $\mathscr{F}$ is a covering of X and if $F\cap A\in\mathscr{F}$ for each $F\in\mathscr{F}$ and each closed subset A of X. A family $\mathscr{G}$ of subsets of X is said to be *$\mathscr{F}$-finite* if each member of $\mathscr{F}$ intersects finitely many members of $\mathscr{G}$. A family $\mathscr{G}$ is said to be *compact-finite* (resp. *cs-finite*) if every compact set (resp. every set which is the closure of a convergent sequence) intersects finitely many members of $\mathscr{G}$. $\mathscr{G}$ is said to be *strongly compact-finite* (resp. *strongly cs-finite*) if the family of closures of members of $\mathscr{G}$ is compact-finite (resp. cs-finite). X is said to be *countably mesocompact* (resp. *strongly countably mesocompact*) if every countable open covering of X has a compact-finite (resp. strongly compact-finite) open refinement. X is said to be *countably sequentially mesocompact* (resp. *strongly countably sequentially mesocompact*) if every countable open covering of X has a cs-finite (resp. strongly cs-finite) open refinement.

THEOREM 1.2. *If a normal space X has W-weak topology with respect to an F-hereditary collection $\mathscr{F}$ and if every countable open covering of X has an $\mathscr{F}$-finite, closed refinement, then X is countably paracompact.*

PROOF. Let $\mathscr{U}$ be any countable, open covering of X. There exists an $\mathscr{F}$-finite, closed refinement $\mathscr{V}$ of $\mathscr{U}$. Since X has W-weak topology with respect to $\mathscr{F}$, therefore $\mathscr{V}$ is locally finite. It follows that $\mathscr{U}$ has a locally finite closed refinement $\mathscr{V}$. We may construct now a locally finite, countable, closed refinement of $\mathscr{U}$ from $\mathscr{V}$. Since X is normal therefore X is countably paracompact.

COROLLARY 1.1. *If a normal space X has W-weak topology with respect to an F-hereditary collection $\mathscr{F}$ and if every countable, open covering of X has an $\mathscr{F}$-finite, open refinement, then X is countably paracompact.*

PROOF. Let $\mathfrak{A}$ be any countable, open covering of X. Let $\mathscr{B}=\{B_\lambda : \lambda\in\Lambda\}$ be an $\mathscr{F}$-finite, open refinement of $\mathfrak{A}$. Since X has W-weak topology with respect to $\mathscr{F}$, therefore $\mathscr{B}$ must be point-finite. X being normal, $\mathscr{B}$ is shrinkable and so there exists an open covering $\mathscr{C}=\{C_\lambda : \lambda\in\Lambda\}$ of X such that $C_\lambda\subset B_\lambda$ for all $\lambda\in\Lambda$, $\{\bar{C}_\lambda : \lambda\in\Lambda\}$ is then an $\mathscr{F}$-finite, closed refinement of $\mathfrak{A}$ and hence X is countably paracompact by the above theorem.

COROLLARY 1.2. *A normal k-space is countably paracompact if and only if every countable, open covering of X has a compact-finite closed refinement.*

PROOF. If X is normal, countably paracompact, then every countable open covering of X has a locally finite closed and hence also a compact-finite closed refinement. Conversely, if every countable open covering of X has a compact-finite closed refinement, then it has also a locally finite closed refinement. Since a space is a k-space if and only if it has W-weak topology with respect to the F-hereditary collection of all compact sets, therefore the result follows from theorem 1.2

Since a space is a sequential space if and only if it has W-weak topology with respect to the F-hereditary collection of all sets which are closures of convergent sequences, therefore in the same manner as above, we have the following:

COROLLARY 1.3. *A normal, sequential space is countably paracompact if and only if every countable open covering of X has a cs-finite, closed refinement.*

THEOREM 1.3. *A locally compact space X is countably paracompact if and only if X is countably mesocompact.*

PROOF. It is easily verified that in a locally compact space, every compact-finite family is locally finite.

THEOREM 1.4. *A first-axiom space is countably paracompact if and only if it is countably sequentially mesocompact.*

PROOF. Every cs-finite family in a first axiom space is locally finite.

THEOREM. 1.5 *A k-space is countably paracompact if and only if it is strongly countably mesocompact.*

PROOF. It is easy to verify that every locally finite family is strongly compact-finite. This proves the "only if" part. To prove the "if" part, let $\mathscr{G}$ be any countable, open covering of X. Let $\mathscr{H}$ be a strongly compact-finite, open refinement of $\mathscr{G}$. Now, X is a k-space and has therefore W-weak topology with respect to the F-hereditary collection of all compact subsets of X. Then the family of closures of members of $\mathscr{H}$ is a compact-finite, closed family and is therefore locally finite. $\mathscr{H}$ is thus locally finite, open refinement of $\mathscr{G}$ and X is therefore countably paracompact.

THEORFM. 1.6 *A sequential space is countably paracompact if and only if it is strongly countably sequentially mesocompact.*

PROOF. Every countably paracompact space is strongly countably sequentially mesocompact since every locally finite family is strongly cs-finite. Conversely, let X be a strongly countably sequentially mesocompact, sequential space. Then, X has W-weak topology with respect to the F-hereditary collection of all sets which are closures of convergent sequences. If $\mathscr{G}$ be any open covering of X and if $\mathscr{H}$ be a strongly cs-finite open refinement, then the family of closures of members of $\mathscr{H}$ is a cs-finite family of closed sets and is therefore locally finite. Thus $\mathscr{H}$ is a locally finite, open refinement of $\mathscr{G}$ and hence X is countably paracompact.

THEOREM 1.7 *For a normal space X, the following are equivalent*:

(a) *X is countably paracompact.*

(b) *X is strongly countably mesocompact.*

(c) *X is countably mesocompact.*

(d) *X is strongly countably sequentially mesocompact.*

(e) *X is countably sequentially mesocompact.*

PROOF. (a)$\Rightarrow$(b)$\Rightarrow$(c) and (d)$\Rightarrow$(e) is immediate since every locally finite family is strongly compact-finite, every strongly compact-finite family is compact-finite and every strongly cs-finite family is cs-finite.

(c)⇒(d). Let $\mathscr{G}=\{G_i:i\in N\}$ be any countable, open covering of X. Since X is countably mesocompact, there exists a compact-finite open refinement $\mathscr{H}=\{H_\lambda:\lambda\in\Lambda\}$ of $\mathscr{G}$. Since $\mathscr{H}$ is compact-finite and hence also point-finite, and X is normal, therefore there exists another open covering $\mathscr{K}=\{K_\lambda:\lambda\in\Lambda\}$ of X such that $\overline{K}_\lambda\subset H_\lambda$ for each $\lambda\in\Lambda$. Now, $\mathscr{H}$ being campact-finite, $\{\overline{K}_\lambda:\lambda\in\Lambda\}$ is compact-finite and hence also cs-finite. Thus $\mathscr{K}$ is a strongly cs-finite open refinement of $\mathscr{G}$ and hence X is strongly countably sequentially mesocompact.

(e)⇒(a). Let $\mathscr{G}=\{G_i:i\in N\}$ be any countable, open covering of X. By hypothesis, there exists a cs-finite, open refinement $\mathscr{H}$ of $\mathscr{G}$. If $x\in X$ and M be the closure of the convergent, constant sequence $\langle x\rangle$ then $x\in M$ and since $\mathscr{H}$ is cs-finite, x can at most belong to finitely many members of $\mathscr{H}$ and hence $\mathscr{H}$ is point-finite. Thus every countable, open covering of X has a point-finite open refinement and hence X is countably paracompact. This cmpletes the proof of the theorem.

Next, we prove two sufficient conditions for a space to be countably paracompact. For this we need the the definition of linearly locally finite families due to H. Tamano [8] and that of order locally finite families due to Y. Katuta [2]. It may be pointed out that after these theorems were obtained, Singal and Arya [5] have recently shown that these theorems hold for several important classes of spaces.

DEFINITION 1.3. A family $\mathscr{U}$ of subsets of a space X equipped with a linear-order '$<$' is said to be *linearly locally finite* if for each $U\in\mathscr{U}$, the family $\{V: V<U\}$ is locally finite. The family $\mathscr{U}$ is said to be *order locally finite* if for each $U\in\mathscr{U}$, the family $\{V:V<U\}$ is locally finite at each point of U.

THEOREM 1.8. *Let $\mathscr{G}$ be an order locally finite open covering of a space X such that $\overline{G}$ is countably paracompact and normal for each $G\in\mathscr{G}$. Then X is countably paracompact and normal.*

PROOF. Let $\mathscr{G}=\{G_\alpha:\alpha\in\Lambda\}$ where Λ is a linearly-ordered index set. For each $\alpha\in\Lambda$, let $F_\alpha=\overline{G}_\alpha\sim\bigcup\{G_\beta:\beta<\alpha\}$. Let $\mathscr{F}=\{F_\alpha:\alpha\in\Lambda\}$. Then $\mathscr{F}$ is a family of closed subsets of X. We shall show that $\mathscr{F}$ is a locally finite closed covering of X. Let $x\in X$. Since $\mathscr{G}$ is a covering of X, there exists $\alpha_0\in\Lambda$ such that $x\in G_{\alpha_0}$

Since $\mathscr{G}$ is order locally finite, $\{G_\alpha : \alpha < \alpha_0\}$ is locally finite at each point of G_α and hence at x. It follows that there exists an open set M_x such that $x \in M_x$ and M_x intersects at most finitely many G_α for $\alpha < \alpha_0$ say G_{α_1}, $G_{\alpha_2}, \cdots\cdots G_{\alpha_n}$, where $\alpha_1 < \alpha_2 < \cdots\cdots < \alpha_n$. In case $M_x \cap G_\alpha = \phi$ for all $\alpha < \alpha_0$, then obviously $x \in F_{\alpha_0}$. Otherwise $x \in F_{\alpha_1}$. Thus $\mathscr{F}$ is a covering of X. Now $M_x \cap G_\alpha = \phi \Rightarrow M_x \cap \overline{G}_\alpha = \phi$. This means that $M_x \cap F_\alpha = \phi$ for all $\alpha < \alpha_0$ except finitely many. Also, $G_{\alpha_0} \cap F_\alpha = \phi$ for all $\alpha < \alpha_0$ by construction. It follows that $M_x \cap G$ is a neighbourhood of x which intersects at most finitely many members of $\mathscr{F}$ and hence $\mathscr{F}$ is locally finite. Thus $\mathscr{F}$ is a locally finite closed covering of X. Also, each F_α is a relatively closed subset of the countably paracompact and normal set $\overline{G}_\alpha$. It follows that $\mathscr{F}$ is a locally finite closed covering of X by countably paracompact and normal sets. Since every space has weak topology with respect to every locally finite, closed covering, therefore X has weak topology with respect to $\mathscr{F}$ and hence X is countably paracompact and normal in view of a result proved by Morita [4].

COROLLARY 1.4. *If $\mathscr{G}$ is a linearly locally finite or a σ-locally finite open covering of a space X such that G is countably paracompact and normal for each $G \in \mathscr{G}$, then X is countably paracompact and normal.*

PROOF. Every σ-locally finite family, as also every linearly locally finite family, is order locally finite.

THEOREM 1.9. *If X is a regular space and if $\mathscr{G}$ is an order locally finite open covering of X such that each $G \in \mathscr{F}$ is countably paracompact, and normal and the frontier of each member of $\mathscr{G}$ is compact, then X is countably paracompact and normal.*

PROOF. Let $G \in \mathscr{G}$ and let F be the frontier of G. Then $F = \overline{G} \sim G$. Since F is compact and the space is regular, there exists a family $\{V_i : i = 1, \cdots, n\}$ of finitely many open subsets of X such that $F \subset \bigcup \{V_i : i = 1, 2, \cdots, n\}$ and each $\overline{V}_i$ is contained in some member of $\mathscr{G}$. For each $i = 1, 2, \cdots, n$, let $M_i = \overline{V}_i \cap \overline{G}$. Then $\mathscr{M} = \{M_i : i = 1, \cdots\cdots, n\} \cup (\overline{G} \sim \bigcup \{V_i : i = 1, 2, \cdots, n\})$ is a finite covering of $\overline{G}$ be relatively closed subsets of $\overline{G}$. Since every closed subspace of a countably paracompact, normal space is countably paracompact and normal. It follows that $\overline{G}$ is countably paracompact and normal as in the proof of theorem 1.8. The result is now obvious in view of theorem 1.8.

COROLLARY 1.5. *If X is a regular space and if $\mathcal{G}$ is a linearly locally finite or a σ-locally finite open covering of X such that each $G\in\mathcal{G}$ is countably paracompact and normal and frontier of each member of $\mathcal{G}$ is compact, then X is countably paracompact and normal.*

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