LATTICE STRUCTURE OF GENERAL TOPOLOGICAL EXTENSIONS

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§ 0. Introduction.

In [7], the author has characterised the epireflection $\beta_E X$ of a space $X$ in an epireflective full subcategory $E$ of the category $T2$ of all Hausdorff spaces, as the space of 'the largest imagive determining type' of nets in $X$ modulo a natural equivalence, topologised in a natural way. In this paper we define $E$-extensions of $E$-regular spaces and study the lattice structure of the collection of all $E$-extensions under a natural order. They form an upper-complete semi-lattice. But further study is rather difficult in such a general set up. So we put some restrictions on the property $E$ (we do not distinguish between 'properties' and 'full subcategories') and/or on the spaces for which $E$-extensions are sought. We introduce the notion of generating families $gE$ or $E$ and study them in detail. In particular, we obtain certain partial results to the problem of when one-point-$E$-extensions exist. We also give certain equivalent forms of being hereditarily $E$ when $E$ has a strongly hereditary pseudoconvergent determining type of nets.

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CONVENTION. $E$ is a full subcategory of $T2$ which is also assumed to be epireflective unless otherwise stated. All spaces considered are objects of $T2$.

§ 1. Preliminaries.

1.1. DEFINITION. A space $X$ is said to be $E$-regular if it is homeomorphic to a subspace of a product of spaces in $E$.

1.2. DEFINITION. Let $X$ be a space. If $Y$ is in $E$ such that $X$ is homeomorphic to a dense subset of $Y$, we say that $Y$ is an $E$-extension of $X$.

1.3. DEFINITION. Let $E$ be a subcategory of $T2$ not necessarily epireflective. Let $NE$ associate to each space $X$, a class of nets $NE(X)$ in $X$ such that $X$ has property $E$ if and only if every net in $NE(X)$ converges to some point in $X$. 
Then $NE$ is called a determining type of nets for $E$ or that $E$ is determined by $NE$.

REMARK. Determining types of nets have been extensively studied by the author in [7]. We quote the following results from [7].

1.4 RESULT. Let $E$ be a full subcategory of $T2$. There exists a determining type of nets $NE$ for $E$ if and only if the empty space as well as the singleton space has $E$.

1.5. DEFINITION. The type of nets $N$ is called imaginary if whenever $f : X \rightarrow Y$ is continuous, $f(N(X)) \subseteq N(Y)$, where $f(N(X)) = \{f \circ s | s \in N(X)\}$.

1.6 RESULT. A full subcategory $E$ of $T2$ is epireflective if and only if there exists an imaginary determining type of nets $NE$ for $E$.

NOTE. $NE(X) = \{s | s$ is a net in $X$ such that if $Y$ is in $E$ and $f : X \rightarrow Y$ any continuous map, then $f \circ s$ converges in $Y\}$, is 'the largest imaginary determining type of nets' for an epireflective subcategory $E$ of $T2$.

1.7 REMARK. In [7], the author has proved that the epireflection $\beta_E(X)$ is the space of equivalence classes of $NE(X)$ topologised in a natural way.

§ 2. $E$-extensions of $E$-regular spaces and generating families for $E$.

2.1 DEFINITION. Let $X$ be an $E$-regular space and let $aX, a'X$ be $E$-extensions of $X$. Then we say that $aX \leq a'X$ if there exists a continuous map from $a'X$ into $aX$, with identity on $X$.

2.2 THEOREM. The collection of all $E$-extensions of an $E$-regular space $X$ forms a complete upper semi-lattice under the partial order generated by the pre-order defined above.

PROOF. The proof is analogous to that in [1] for the corresponding theorem for compactifications.

Let $\{a_iX\}_{i \in J}$ be a family of $E$-extensions of $X$. To show that there exists an $E$-extension $aX$ of $X$ such that $aX = \bigvee_i a_iX$. Let $f : X \rightarrow \prod_i a_iX$ be defined as $f(x) = (a_i(x))_{i \in J}$ where $a_i(x)$ denotes the image of $x$ in $a_iX$ under the homeomorphic embedding. It can be easily checked that $f$ is a homeomorphism(cf. [5] Theorem 2.1). Now $clf(X) = aX$ is an $E$-extension, being a closed subspace of a product of spaces in $E$. Further $aX \geq a_iX$ for each $i$ in $J$ since the projection from $aX$
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onto \( a_iX \) is continuous and is identity on \( X \). Also if \( a'iX \) is an \( E \)-extension of \( X \) such that \( a'iX \geq a_iX \) for every \( i \), then \( a'iX \geq aX \) since the product continuous function \( II_i f_i \) suffices where \( f_i : a'iX \to a_iX \). Thus \( aX = \bigvee a_iX \).

NOTE. \( \beta E X \), the epireflection of \( X \) in \( E \) is the largest \( E \)-extension in this order. In particular, when \( X \) has \( E \), then \( X \) is its own largest \( E \)-extension.

REMARK. To get more information about the semilattice in this most general set up is rather difficult. So we introduce certain restrictions on the property \( E \) and/or on the spaces for which \( E \)-extensions are sought. We introduce the notion of a generating family for the property \( E \).

2.3 DEFINITION. A collection of spaces \( (Y_i)_{i \in I} \) is called a generating family for \( E \) if the following happens: a space \( X \) has \( E \) if and only if \( X \) is homeomorphic to a closed subspace of a product of spaces \( (Y_i)_{i \in K}, K \subseteq J \). If there exists a finite (respectively singleton) generating family for \( E \) then \( E \) is said to be finitely (respectively singly) generated.

EXAMPLE. (i) If \( I \) is the closed unit interval of reals, then \( (I) \) is a generating family for compactness.

(ii) If \( J \) is the open unit interval of reals, then \( (J) \) is a generating family for realcompactness.

(iii) The discrete dyad \( D \) forms a generating family \( (D) \) of being zero dimensional and compact.

NOTE. It is not hard to see that a property \( E \) is finitely generated if and only if singly generated. Some singly generated epireflective full subcategories of \( T2 \) are studied by S. Mrowka in [5].

2.4 RESULT. The generating families for a property \( E \) form an upper complete semilattice under inclusion. This semilattice possesses minimal elements if and only if \( E \) is singly generated. It is not necessarily a lattice even when \( E \) is singly generated.

The first two statements can be proved easily. To justify the last statement, consider \( E=\text{realcompactness} \). If \( J=(0,1) \), then it is known that \( (J) \) is a generating family for \( E \). It can be fairly easily proved that if \( K=[0,1] \), then \( (K) \) is also a generating family for \( E \). \( (J) \cap (K) = \emptyset \) is clearly not a generating family.

CONVENTION. In the rest of the paper \( gE \) stands for a typical generating family.
for $E$.

2.5 REMARK. Let $E$ be a closed hereditary productive property. If we define $NE(X) = \{s|\text{for any } Y \text{ in } gE \text{ and for any continuous function } f : X \rightarrow Y, f \circ s \text{ is convergent in } Y\}$, then it can be seen that $NE$ is an imagive determining type of nets for $E$. Hence or independently it is seen that $NE$ is precisely $NE_1$, the largest imagive one.

2.6 DEFINITION. A subset $A$ of $X$ is said to be \textit{gE-embedded} in $X$ if every continuous function on $A$ to any space $Y$ in $gE$ extends continuously to $X$.

2.7 RESULT. If a closed subset $A$ is \textit{gE-embedded} in $X$, and $\beta_EX$ is the largest $E$-extension of $X$, then $\text{cl}_{\beta_EX} A = \beta_EA$.

The proof is easy and omitted.

NOTE. The converse of Result 2.7 is not true. Example: $E=\text{realcompactness}$, $gE=(J)$ where $J$ is the open unit interval of reals and $X$ is a non-normal real-compact space.

2.8 DEFINITION. An $E$-regular space $X$ is said to be \textit{gE-normal} if every closed subset of $X$ is \textit{gE-embedded} in $X$.

NOTE. $gE$-normal $\Rightarrow$ normal.

2.9 REMARK. Herrlich, H. [3] has defined a space to be $E$-normal if every disjoint pair of closed subsets in it is $E$-separated, i.e., if whenever $A$ and $B$ are disjoint closed subsets of $X$, there exists an $E$-space $Y$ and a continuous function $f : X \rightarrow Y$ such that $\text{cl}f(A) \cap \text{cl}f(B) = \emptyset$. It can be easily seen that $gE$-normal $\Rightarrow$ $E$-normal for any generating family $gE$ of $E$. Notice that any $E$-space is $E$-normal; but not necessarily $gE$ normal. For example, $R_1 \times R_i$ where $R_i$ is the set of reals with lower limit topology is realcompact and (realcompact)-normal in the sense of Herlich. But it is not $gE$-normal for any $gE$ since it is not normal. Notice that in particular $(R)$-normal if and only if normal. (cf. [2] 3D1 (48)). Thus again, a converse of result 2.7 is not true; however a partial converse can be given as follows:

2.10 RESULT. If $X$ or $\beta_EX$ is $gE$ normal, then whenever $\text{cl}_{\beta_EX} Y = \beta_EY$, we have $Y$ is $gE$-embedded in $X$.

The proof is easy and omitted.

NOTE. Now we come back to the consideration of the collection of $E$-extens-
ions, and in particular, to the problem of existence of one-point-$E$-extensions.

2.11 DEFINITION. A regular space $X$ is called locally $E$ if each point of $X$ has a basis of $E$-neighbourhoods.

2.12 THEOREM. Let $X$ be $E$-regular and $\beta EX$ regular. If $X$ is open in $\beta EX$, then $X$ is locally $E$. On the other hand, if $X$ is locally $E$ and $gE$-normal for some generating family $gE$, then $X$ is open in $\beta EX$.

PROOF. Suppose $X$ is open in $\beta EX$; since $\beta EX$ is regular, it follows that $X$ is locally $E$. Conversely, suppose $X$ is locally $E$. If $X$ has $E$, then trivial, since $\beta EX=X$ in that case. If $X$ does not have $E$, then $X$ is open in $\beta EX$-X converging to a point $p$ in $X$. Then for each $j$ in $D$, $s(j)$ is a net in $NE_i(X)$ converging to a point $s(j)$ in $\beta EX$. Consider the product net $P$ in $X$ corresponding to $s$. This product net $P$ converges to $p$. Take a neighbourhood $V$ of $p$ in $\beta EX-X$ such that $V\cap X$ is closed in $X$ and has $E$. The net $P$ is eventually in $V\cap X$ (say) after $(m_0,f_0)\in D\times \prod_{i\in D}E_i$. Consider $s(m)$ where $m>m_0$. It can be easily seen that this net which is a member of $NE_i(X)$ is eventually in $V\cap X$. But since $X$ is $gE$-normal, by remark 2.5, it follows that $s(m)\in NE_i(V\cap X)$. Now $V\cap X$ has $E$ so that $s(m)$ converges inside $V\cap X$, which is a contradiction. Hence any net in $\beta EX-X$ converges in $\beta EX-X$, if at all it converges. Hence $X$ is open in $\beta EX$.

NOTE. $X$ is open $\beta EX$ does not imply that $X$ is $gE$ normal for any $gE$. For example, suppose $X$ is realcompact and non-normal. Then trivially $X$ is open in $\nu X=X$. But since $X$ is not normal, it is not $gE$-normal for any $gE$ where $E=\text{realcompactness}$.

QUESTION. If $X$ is $E$-regular and locally $E$, $\beta EX$ is regular and if either $X$ or $\beta EX$ is $gE$-normal, for some generator $gE$ of $E$, then does it follow that $X$ is open in $\beta EX$?

2.13 DEFINITION. A property $E$ is collapsible if for every space $X$ which is a proper open subset of $\beta EX$, the identification of $\beta EX-X$ to a point has $E$.

2.14 RESULT. Let $E$ be collapsible. Let $X$ be $E$-regular and $gE$-normal. Suppose $X$ does not have $E$. Further let $\beta EX$ be regular. Then $X$ has a one-point-$E$-
extension, if and only if $X$ is locally $E$.

2.15 REMARK. If $X$ has $E$ and has a one-point-$E$-extension, then $X$ is not $E$-closed. (we call a space $E$-closed, if it is closed in every space with $E$ containing it). On the other hand, if $X$ is not $E$-closed, then a sufficient condition that a one-point-$E$-extension exists is that the union of a compact subset and any subset having $E$ has again $E$. We will describe certain such situations in then next section.

§ 3. Pseudoconvergent determining types of nets and Hereditarily $E$ spaces.

3.1 DEFINITION. A determining type of nets $NE$ is called strongly hereditary if whenever $A \subseteq X$, then $NE(A) \subseteq NE(X)$ and furthermore if $s$ is a net in $A$ such that $s \in NE(X)$ then $s \in NE(A)$.

3.2 DEFINITION. A type of nets $N$ is called pseudoconvergent if every net in $N(X)$ is pseudoconvergent for any $X$.

CONVENTION. In this section we consider epireflective subcategories $E$ of $T2$ such that $E$ has a strongly hereditary pseudoconvergent determining type of nets $NE$.

EXAMPLES. (for details see [6])
(i) $E=$compactness; $NE=$ {Universal nets} or $=\{weakly\ open\ universal\ nets\}$
(ii) $E=\alpha'$-spaces; $NE=\{\sigma$-directed weakly open-universal nets$\}$.
(iii) $E=\alpha''$-spaces; $NE=\{\sigma$-directed strongly closed-universal nets$\}$.

3.3 THEOREM. If $NE$ is an imagine strongly hereditary pseudoconvergent determining type of nets for $E$ and if $\beta_E X$ is regular for an $E$-regular space $X$, then $X$ is open in $\beta_E X$ if and only if $X$ is locally $E$.

Proof easy and omitted.

3.4 THEOREM. If $X=Y \cup Z$ where $Y$ has $E$ and $Z$ is compact, then $X$ has $E$.

PROOF. Let $s \in NE(X)$. If $s$ is frequently in $Z$, then it has a convergent subnet and since $s$ is pseudoconvergent, it follows that $s$ converges. If $s$ is eventually in $cZ \subseteq Y$, then $s \in NE(Y)$ since $NE$ is strongly hereditary. Now since $Y$ has $E$, $s$ is convergent. Hence $X$ has $E$.

3.5. NOTE. See Remark 2.15.

3.6 THEOREM. The following are equivalent on a space $Y$:
(a) $Y$ is hereditarily $E$, i.e., every subspace of $Y$ has $E$.
(b) For each space $X$, if there exists a map $f: X \to Y$ such that $f^{-1}(y)$ is compact
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for each \( y \) in \( Y \), then \( Y \) has \( E \).

(c) Every space of which \( Y \) is a one-one continuous image, has \( E \).

(d) For each point \( y \) in \( Y \), \( Y - \{y\} \) has \( E \).

PROOF. Use Theorem 3.4 and proceed along the same lines as those of Theorem 8.17 in [2] (122).

To show (c) implies (a), the technique employed is the same, by noticing the following: Given a subspace \( Z \subseteq Y \), by enlarging the topology of \( Y \) making \( Z \) and \( Y - Z \) open, the new space obtained is \( E \)-regular.

3.7 COROLLARY. If \( f : X \rightarrow Y \) is one-one and continuous and if \( Y \) is hereditarily \( E \), then \( X \) is hereditarily \( E \).

3.8 NOTE. If every subspace of \( Y \) has \( E \), and if \( \text{card } Y = m \), then every discrete space of cardinality \( \leq m \) has \( E \).

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