A NOTE ON BLOCK CIRCULANT MATRICES

By Chong-Yun Chao*

The purpose of this note is to present a simple proof for the main theorem in [3]. Our method is similar to the one used in [1].

Let $C=(c_{ij})$ be a $n\times n$ circulant with $c_{ij}$ belonging to the complex number field $K$, and $i,j=0,1,\ldots,n-1$. Let $f(x)=\sum_{j=0}^{n-1} c_{ij}x^j$. It is well known that the eigenvalues of $C$ are $\mu_k=f(\omega^k)$, $k=0,1,\ldots,n-1$, and the eigenvector corresponding to each $\mu_k$ is a column vector $\omega^k=(\omega^k, \omega^{2k}, \omega^{3k}, \ldots, \omega^{(n-1)k})$ for $k=0,1,\ldots,n-1$ where $\omega=\exp\{2\pi i/n\}$ (in fact, $\omega$ can be any primitive $n$-th root of unity). Let $P=(p_{ij})=\omega^{ij}$, $i,j=0,1,\ldots,n-1$, $Q=(1/\sqrt{n})P$. Then $P$ is a Vandermonde matrix and $Q$ is a unitary matrix.

Also,

$$Q^{-1}CQ=Q'CQ=\text{diag} \{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$$

where $\text{diag} \{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$ denotes the diagonal matrix with $\mu_0, \mu_1, \ldots, \mu_{n-1}$ on the diagonal, and $Q'$ and $Q$ denote the transpose and complex conjugate of $Q$ respectively.

The main theorem (Theorem 5) in [3] states as follows: Let

$$A=\begin{bmatrix}
A_0 & A_1 & \cdots & A_{m-1} \\
A_{m-1} & A_0 & \cdots & A_{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \cdots & A_0
\end{bmatrix}$$

be a $m\times m$ block circulant with each $A_i$ being a $n\times n$ circulant matrix. Let $P=(p_{ij})=\omega^{ij}$ be the $n\times n$ matrix as before. Let $r_0, r_1, \ldots, r_{m-1}$ be the $m$-th roots of unity.

If $Q$ is given by the following matrix:

---

*This work was done while the author was at Carnegie-Mellon University (under a faculty exchange program between Carnegie-Mellon University and the University of Pittsburgh). The author wishes to thank Professor R. J. Duffin for helpful discussions.
We have $Q^{-1}AQ=D$ with $D$ being a matrix of diagonal blocks $D_0$, $D_1$, ..., $D_{m-1}$ where each $D_i$ is diagonal. The diagonal elements are given by the eigenvalues of the matrix $\sum_{k=0}^{m-1} A_k r^k$. Moreover, given any $nm \times nm$ diagonal block matrix $D = \text{diag} \{D_0, D_1, \ldots, D_{m-1}\}$ where each $D_i$ is a $n \times n$ diagonal matrix, $A = QDQ^{-1}$ is a block circulant with each block being a circulant matrix.

We note that $r_0, r_1, \ldots, r_{m-1}$ are $m$-th roots of unity implying $r_j = r^j$, $j = 0$, 1, ..., $m-1$ where $r = \exp \{2\pi i/m\}$.

In [3], a proof for three block case ($m=3$) is given, and it states that the proof for the general case is omitted since it is just an extension of the three block case. Here we present a simple proof for the general case by using elementary properties of Kronecker product of matrices.

The proof goes as follows: The matrix $A$ is equal to

$$I \otimes A_0 + T \otimes A_1 + T^2 \otimes A_2 + \cdots + T^{m-1} \otimes A_{m-1}$$

where $T$ is the $m \times m$ permutation matrix corresponding to the permutation

$$
\begin{pmatrix}
0 & 1 & \cdots & i & \cdots & m-1 \\
1 & 2 & \cdots & i+1 & \cdots & 0
\end{pmatrix}
$$

$I = T^m$ is the identity matrix and $\otimes$ denotes the Kronecker product. Clearly, each $T^k$, $k = 0$, 1, ..., $m-1$, is a $m \times m$ circulant.

Let $R = (s_{ij}) = (r^{ij})$ be a $m \times m$ matrix with $r = \exp \{2\pi i/m\}$ and $\Gamma = (1/\sqrt{m})R$. Then, again, $R$ is a $m \times m$ Vandermonde matrix and $\Gamma$ is a unitary matrix. By using elementary properties of Kronecker product of matrices (e.g., see pp.68-70 in [2]), we have

$$(\Gamma \otimes \Omega)^{-1} A (\Gamma \otimes \Omega)$$

$$= (\Gamma^{-1} \otimes \Omega^{-1})(I \otimes A_0 + T \otimes A_1 + \cdots + T^{m-1} \otimes A_{m-1})(\Gamma \otimes \Omega)$$

$$= (I \otimes \Omega^{-1} A_0 \Omega) + (\Gamma^{-1} T \Gamma \otimes \Omega^{-1} A_1 \Omega) + \cdots + (\Gamma^{-1} T^{m-1} \Gamma \otimes \Omega^{-1} A_{m-1} \Omega)$$

(1)

Since each $A_k$ is a $n \times n$ circulant, $Q^{-1} A_k \Omega$ is a diagonal matrix, denoted by
$E_k$, with eigenvalues of $A_k$ on the diagonal for $k=0, 1, \ldots, n-1$. Since each $T^j$ is a $m \times m$ permutation matrix, $\Gamma^{-1}T^j\Gamma = \text{diag} \{ r^0, r^1, r^2, \ldots, r^{(m-1)j} \}$ for $j=0, 1, \ldots, m-1$. Thus, (1) is equal to, i.e.,

$$(\Gamma \otimes \Omega)^{-1}A(\Gamma \otimes \Omega)$$

$$= I \otimes E_0 + \text{diag} \{ r^0, r^1, \ldots, r^{m-1} \} \otimes E_1 + \cdots + \text{diag} \{ r^0, r^{m-1}, \ldots, r^{(m-1)(m-1)} \} \otimes E_{m-1}$$

$$= \text{diag} \left\{ \sum_{k=0}^{m-1} E_k, \sum_{k=0}^{m-1} r^k E_k, \sum_{k=0}^{m-1} r^{2k} E_k, \ldots, \sum_{k=0}^{m-1} r^{(m-1)k} E_k \right\}.$$

This means that the $i$-th diagonal element is a diagonal matrix denoted by $D_i$, and the diagonal elements of $D_i$ are given by the eigenvalues of the matrix $\sum_{k=0}^{m-1} A_i r^{ik}$.

Now we show that $A=(\Gamma \otimes \Omega)D(\Gamma \otimes \Omega)^{-1}$ is a block circulant with each block being a circulant matrix and $D=\text{diag} \{ D_0, D_1, \ldots, D_{m-1} \}$ where each $D_i$ is an $n \times n$ diagonal matrix. We need the following:

**LEMMA.** If $D_p=\text{diag} \{ d^p_0, d^p_1, \ldots, d^p_{(n-1)(n-1)} \}$ then $F_p=QD_pQ^{-1}$ is a circulant for $p=0, 1, \ldots, m-1$.

**PROOF.**

$$\left( QD_p Q^{-1} \right)_{ij} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{ik} d^p_k \omega^{jk},$$

$$\left( QD_p Q^{-1} \right)_{(i+1)(j+1)} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{(i+1)k} d^p_k \omega^{(j+1)k},$$

since $\omega \bar{\omega}=1$, $(QD_p Q^{-1})_{ij}=(QD_p Q^{-1})_{(i+1)(j+1)}$ for $i, j=0, 1, \ldots, n-1$ (the subscripts are taken modulo $n$).

Hence each $F_p=QD_pQ^{-1}$ is a circulant for $p=0, 1, \ldots, m-1$.

Now, by using our Lemma,

$$A=(\Gamma \otimes \Omega)D(\Gamma \otimes \Omega)^{-1}$$

$$= (\Gamma \otimes \Omega) \left( \text{diag} \{ D_0, D_1, \ldots, D_{m-1} \} \right) \left( (\Gamma^{-1} \otimes \Omega)^{-1} \right)$$

$$= (\Gamma \otimes \Omega) \left[ \text{diag} \{ 1, 0, \ldots, 0 \} \otimes D_0 + \text{diag} \{ 0, 1, 0, \ldots, 0 \} \otimes D_1 + \cdots \right.$$

$$+ \text{diag} \{ 0, 0, \ldots, 0, 1 \} \otimes D_{m-1} \right] \left( (\Gamma^{-1} \otimes \Omega)^{-1} \right)$$

$$= \Gamma \left( \text{diag} \{ 1, 0, \ldots, 0 \} \right) \Gamma^{-1} \otimes F_0 + \Gamma \left( \text{diag} \{ 0, 1, 0, \ldots, 0 \} \right) \Gamma^{-1} \otimes F_1 + \cdots$$

$$+ \Gamma \left( \text{diag} \{ 0, 0, \ldots, 0, 1 \} \right) \Gamma^{-1} \otimes F_{m-1}. \quad (2)$$

Hence, by using (2), the $ij$ entry in the matrix $A$ is
and the \((i+1) \ (j+1)\) entry in the matrix \(A\) is
\[
\frac{1}{m} (r^{i+1}P_{0}^{j+1}F_{0} + \cdots + r^{(m-1)}P_{0}^{j+1}F_{m-1}),
\]
Since \(r F = 1\), the \(ij\) entry in \(A\) is equal to the \((i+1) \ (j+1)\) entry in \(A\) for \(i, j = 0, 1, \ldots, m-1\) (all the \(ij\) entries are taken modulo \(m\)). Hence, \(A\) is a block circulant.

The matrix \(Q\) in the theorem is our \(R \otimes P\).

Then \(Q^{-1} = \frac{1}{mn} (R^{-1} \otimes P^{-1})\), and
\[
Q^{-1}A = \frac{1}{mn} (R^{-1} \otimes P^{-1}) A (R \otimes P)
= \left(\frac{1}{\sqrt{mn}} (R^{-1} \otimes P^{-1})\right) A \left(\frac{1}{\sqrt{mn}} (R \otimes P)\right)
= (G \otimes Q)^{-1} A (G \otimes Q).
\]

REMARKS. It is indicated on page 19 in [3] that \(R\) and \(P\) are Vandermonde matrices and the determinant of a Vandermonde matrix is well known. However, \(Q\) is not a Vandermonde matrix. But the determinant of \(Q\), \(\det Q\), is equal to \(\det (R \otimes P) = (\det R)^{m} (\det P)^{m}\) (see p. 70 in [2]). G. Trapp pointed out to me that, using the same method, one can deal with a block circulant each whose entry is again a block circulant, \(\ldots\), etc.