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# THE LOGARITHMIC (L) MEAN OF THE DIFFERENTIATED DOUBLE FOURIER SERIES

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**1.** The double Fourier series corresponding to $2\pi$-periodic function $f(x, y) \in L[-\pi, \pi; -\pi, \pi]$ is given by

(1.1) $$f(x,y) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}(x,y)$$

where

$$A_{mn} = \lambda_{mn}(a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny)$$

with the usual meanings of $\lambda_{mn}$, $a_{mn}$ etc.

We write (1.1) as

(1.1′) $$f(x,y) \sim \sum_{1}^{\infty} \sum_{1}^{\infty} (a,\ b,\ c,\ d;\ x,\ y)_{mn}$$

Differentiating (1.1′) once with respect to $x$, once with respect to $y$ and successively with respect to $x$ and $y$, we get the following three allied or conjugate series after ignoring the numerical coefficients so introduced

(1.2) $$\sum_{1}^{\infty} \sum_{1}^{\infty} (d,\ -c,\ -b,\ a;\ x,\ y)_{mn} = \sum_{1}^{\infty} \sum_{1}^{\infty} B_{mn}(x,\ y)$$

(1.3) $$\sum_{1}^{\infty} \sum_{1}^{\infty} (-b,\ -a,\ d,\ -c;\ x,\ y)_{mn} = \sum_{1}^{\infty} \sum_{1}^{\infty} C_{mn}(x,\ y)$$

(1.4) $$\sum_{1}^{\infty} \sum_{1}^{\infty} (c,\ d,\ -a,\ -b;\ x,\ y)_{mn} = \sum_{1}^{\infty} \sum_{1}^{\infty} D_{mn}(x,\ y)$$

We call the series (1.2), (1.3) and (1.4) the first, second and the third allied series respectively. But if we retain the numerical coefficients these are

(1.2′) $$\sum_{1}^{\infty} \sum_{1}^{\infty} mn\ B_{mn}$$

(1.3′) $$\sum_{1}^{\infty} \sum_{1}^{\infty} m\ C_{mn}$$

(1.4′) $$\sum_{1}^{\infty} \sum_{1}^{\infty} n\ D_{mn}$$

Let us write

$$\phi(u, v)=\frac{1}{4}[f(x+u, y+v)+f(x-u, y+v)+f(x+u, y-v)+f(x-u, y-v)]$$
$$\sim\sum_{1}^{\infty}\sum_{1}^{\infty} A_{mn}\cos mx\cos ny$$

$$\psi(u, v)=\frac{1}{4}[f(x+u, y+v)-f(x-u, y+v)-f(x+u, y-v)+f(x-u, y-v)]$$
$$\sim\sum_{1}^{\infty}\sum_{1}^{\infty} B_{mn}\sin mu\sin nv$$

$$\phi_1(u, v)=\frac{1}{4}[f(x+u, y+v)+f(x+u, y-v)-f(x-u, y+v)-f(x-u, y-v)]$$
$$\sim\sum_{1}^{\infty}\sum_{1}^{\infty} C_{mn}\sin mu\cos nv$$

$$\phi_2(u, v)=\frac{1}{4}[f(x+u, y+v)-f(x+u, y-v)+f(x-u, y+v)-f(x-u, y-v)]$$
$$\sim\sum_{1}^{\infty}\sum_{1}^{\infty} D_{mn}\cos mu\sin nv$$

The conjugate function's associated with the series (1.2), (1.3) and (1.4) are respectively

$$\bar{f}(x,y)=\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{1}{\pi^2}\int_{\varepsilon_1}^{\pi}\int_{\varepsilon_2}^{\pi}\psi(u, v)\cot u/2\cot v/2\,dudv$$

$$\bar{f}_1(x,y)=\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{1}{\pi^2}\int_{\varepsilon_1}^{\pi}\int_{\varepsilon_2}^{\pi}\phi_1(u, v)\cot u/2\,dudv$$

$$\bar{f}_2(x,y)=\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{1}{\pi^2}\int_{\varepsilon_1}^{\pi}\int_{\varepsilon_2}^{\pi}\phi_2(u, v)\cot v/2\,dudv$$

provided the limits exist.

DEFINITION. A double series $\sum\sum u_{mn}$, with partial sums $S_{mn}$ is said to be *summable by logarithmic method of summability* $(L)$ *to the sum* $S$, if for $q, r\in(0,1)$,

$$\lim\{\log(1-q)\log(1-r)\}^{-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{S_{mn}q^m r^n}{mn}=S$$

as $q, r\to 1-0$. We write it as $\sum\sum U_{mn}=S(L)$

**2.** Corresponding to the $2\pi$-periodic and Lebesgue integrable function $f(x)$, the Fourier series and its conjugate series are respectively given by

(2.1) $$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum A_n(x)$$

(2.2) $$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum B_n(x)$$

The derived series of (2.1) is

(2.3) $$\sum n B_n(x)$$

Adopting the notations

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s\}$$

$$\psi(t) = f(x+t) - f(x-t)$$

$$h(t) = \psi(t) - D, \quad g(t) = \frac{\psi(t)}{4 \sin t/2} - C$$

Mohanty and Nanda [1] proved the following theorems:

THEOREM A. If

$$\int_t^{\pi} \frac{h(u)}{u} du = o\left(\log \frac{1}{t}\right) \text{ as } t \to 0^+$$

*then the sequence* $\{nB_n\}$ *is summable (L) to* $D\pi^{-1}$

THEOREM B. *If*

$$\int_t^{\pi} \frac{|g(u)|}{u} du = o\left(\log \frac{1}{t}\right) \text{ as } t \to 0^+$$

*then the series* $\sum_1^{\infty} nB_n$ *is summable (L) to C.*

In another paper Nanda [2] established the following theorems.

THEOREM C. *If*

$$\int_t^{\pi} \frac{\phi(u)}{u} du = o\left(\log \frac{1}{t}\right) \text{ as } t \to 0^+$$

*then the series (2.1) i.e.* $\sum A_n$ *is summable (L) to the value S.*

THEOREM D. *If*

$$\int_t^{\pi} \frac{g(u)}{u} du = o\left(\log \frac{1}{t}\right) \text{ as } t \to 0^+$$

*then the series (2.3) i.e.* $\sum_1^{\infty} n B_n$ *is summable (L) to the value C.*

Clearly the theorem D is an improvement upon the theorem B.
Nanda and Das [3] have further improved the theorems A, C and D.

3. Regarding $(L)$ summability of double Fourier series (1.1), Singh [4] has established the two variable analogue of the theorem C. Also the present author [5] has extended the theorem 1 of the paper [3] to the case of double Fourier series. The object of the present paper is to generalise the theorems A and D to the case of the two variables in the form of the theorems A′ and D′ to be given below.

Let us define

$$h(u,v)=\phi(u,v)-D,\quad g(u,v)=\frac{\phi(u,v)}{16\sin u/2\sin v/2}-C$$

where $D$ and $C$ are functions of $x$ and $y$.

In fact we propose to establish the theorems:

THEOREM A′. *If*

$$\int_s^\pi\int_t^\pi \frac{h(u,v)}{uv}dudv=o\left(\log\frac{1}{s}\cdot\text{lo}\ \frac{1}{t}\right) \tag{3.1}$$

*as* $s,t\to 0^+$, *then the sequence* $\{mn\,B_{mn}\}$ *is summable* $(L)$ *to* $D\pi^{-2}$.

THEOREM D′. *If*

$$G(s,t)=\int_s^\pi\int_t^\pi \frac{g(u,v)}{uv}dudv=o\left(\log\frac{1}{s}\cdot\log\frac{1}{t}\right) \tag{3.2}$$

*as* $s,t\to 0^+$, *then the series* (1.2) *i.e.* $\sum\sum mn\,B_{mn}$ *is summable* $(L)$ *to the value* $C$.

4. We require the following results for the proof of the theorem A′.

(a) Let $\xi=\arc\sin(1-q)$, $\eta=\arc\sin(1-r)$

$$\Delta=1-2r\cos t+r^2$$

$$Q(r,t)=Q(t)=r\sin t/\Delta,\quad \rho(r,t)=\rho(t)=\frac{1}{2}\cot t/2-Q(t)$$

so that

$$\frac{dQ}{dt}=Q'(t)=r\{(r+r^2)\cos t-2r\}/\Delta^2$$

$$\frac{d\rho}{dt}=\rho'(t)=-(1-r)^2\text{cosec}^2 t/2/4\Delta-2r(1-r)^2\cos^2 t/2/\Delta^2$$

It is known that [1], [6]

$$\begin{cases}Q(\pi)=0, & Q(\eta)=o\left(\frac{1}{\eta}\right)\\ \rho(\pi)=0, & \rho(\eta)=o(1/\eta)\end{cases}$$

$$\begin{cases}Q'(t)=o(1/\eta^2) & (0<t\le\eta)\\ \rho'(t)=o\{(1-r)^2/t^4)\}=o(\eta^2/t^4)\end{cases}$$

(b) Let

$$\Psi(s,t)=\int_0^s\int_0^t\phi(u,v)dudv$$

Evidently with $h(u,v)=\phi(u,v)$

$$(3.1)\Rightarrow[\Psi(s,t)=o\left(st\log\frac{1}{s}\cdot\log\frac{1}{t}\right) \tag{4.3}$$

Proof of theorem A′-without affecting the generality of the theorem A′, we may as in [3] assume $D=0$. In such a case

$$h(u,v)=\phi(u,v)$$

The theorem will be proved if we show that

$$\begin{aligned}V(x,y)=V(x,y;\ q,r)&=\sum_1^\infty\sum_1^\infty mnB_{mn}q^m r^n/mn\\&=\sum_1^\infty\sum_1^\infty B_{mn}q^m r^n=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)\end{aligned} \tag{4.4}$$

we have

$$\begin{aligned}V(x,y)&=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi\phi(s,t)\,Q(q,s)\,Q(r,t)\,dsdt\\&=\frac{1}{\pi^2}\left(\int_\xi^\pi\int_\eta^\pi+\int_\xi^\pi\int_0^\eta+\int_0^\xi\int_\eta^\pi+\int_0^\xi\int_0^\eta\right)\\&=\frac{1}{\pi^2}\sum_1^4 I_i,\ \text{say.}\end{aligned}$$

Here

$$\begin{aligned}I_1=\int_\xi^\pi\int_\eta^\pi\phi(s,t)\Big[&\frac{1}{4}\cot s/2\cot t/2-\frac{1}{2}\cot t/2\,\rho(s)\\&-\frac{1}{2}\cot\frac{s}{2}\rho(t)+\rho(s)\rho(t)\Big]dsdt\\=I_{1,1}+&I_{1,2}+I_{1,3}+I_{1,4},\ \text{say.}\end{aligned}$$

where

$$I_{1,1}=\int_\xi^\pi\int_\eta^\pi\phi(s,t)\cot s/2\ \cot t/2\,dsdt$$

$$=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right) \tag{4.6}$$

by (4.3) and noticing that $\cot t/2$ behaves as $\frac{2}{t}$.

$$I_{1,4}=\int_{\xi}^{\pi}\int_{\eta}^{\pi}\phi(s,t)\ \rho(s)\ \rho(t)\ dsdt$$

$$=[\Psi(s,t)\rho(s)\rho(t)]_{\xi,\eta}^{\pi,\pi}-\int_{\xi}^{\pi}[\Psi(s,t)\rho(t)]_{\eta}^{\pi}\rho'(s)ds$$

$$-\int_{\eta}^{\pi}[\Psi(s,t)\rho(s)]_{\xi}^{\pi}\rho'(t)dt+\int_{\xi}^{\pi}\int_{\eta}^{\pi}\Psi(s,t)\rho'(s)\rho'(t)dsdt$$

$$=o\Big(\xi\log\frac{1}{\xi}\cdot\frac{1}{\xi}\int_{\eta}^{\pi}t\log\frac{1}{t}\,o(\eta^2/t^4)dt\Big)+o\Big(\eta\log\frac{1}{\eta}\cdot\frac{1}{\eta}\int_{\xi}^{\pi}s\log\frac{1}{s}\,o(\xi^2/s^4)ds\Big)$$

$$+o\Big(\xi\,\eta\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\cdot\frac{1}{\xi\eta}\Big)+o\Big(\int_{\xi}^{\pi}\int_{\eta}^{\pi}st\log\frac{1}{s}\log\frac{1}{t}\cdot o(\xi^2\eta^2/s^4t^4)dsdt\Big)$$

$$=o\Big(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\Big)$$

$$=o\Big(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\Big) \tag{4.7}$$

$$I_{1,2}=o\Big(\int_{\xi}^{\pi}\int_{\eta}^{\pi}\frac{\phi(s,t)}{st}\,ds\,dt\Big)$$

$$=o\Big(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\Big)=o\Big(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\Big) \tag{4.8}$$

similarly

$$I_{1,3}=o\Big(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\Big) \tag{4.9}$$

From (4.6), (4.7), (4.8) and (4.9), we get

$$I_1=o\Big(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\Big) \tag{4.10}$$

Again integrating by parts [7]

$$I_4=\int_0^{\xi}\int_0^{\eta}\phi(s,t)Q(q,s)Q(r,t)ds\,dt$$

$$=[\Psi(\xi,\eta)Q(\xi)Q(\eta)]-Q(\eta)\int_0^{\xi}\Psi(s,\eta)Q'(s)ds$$

$$-Q(\xi)\int_0^{\eta}\Psi(\xi,t)Q'(t)dt+\int_0^{\xi}\int_0^{\eta}\Psi(s,t)Q'(s)Q'(t)ds\,dt$$

$$=\left[o\left(\xi\eta\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)\cdot o\left(\frac{1}{\xi}\right)\cdot o\left(\frac{1}{\eta}\right)\right]+o\left(\frac{1}{\eta}\right)\int_0^{\xi}o\left(s\eta\log\frac{1}{s}\cdot\log\frac{1}{\eta}\right)$$

$$\times o\left(\frac{1}{s^2}\right)ds+o\left(\frac{1}{\xi}\right)\int_0^{\eta}o\left(\xi t\log\frac{1}{\xi}\cdot\log\frac{1}{t}\right)o\left(\frac{1}{t^2}\right)dt$$

$$+\int_0^{\xi}\int_0^{\eta}o\left(st\log\frac{1}{s}\log\frac{1}{t}\right)\cdot o\left(\frac{1}{s^2}\right)\cdot o\left(\frac{1}{t^2}\right)\cdot o\left(\frac{1}{t^2}\right)dsdt$$

(4.11) $$=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

Also

$$I_2=\int_{\xi}^{\pi}\int_0^{\eta}\phi(s,t)Q(s)Q(t)dsdt$$

$$=[\Psi(s,t)Q(s)Q(t)]_{\xi,0}^{\pi,\pi}-\int_{\xi}^{\pi}[\Psi(s,t)Q(t)]_0^{\eta}Q'(s)ds$$

$$-\int_0^{\eta}[\Psi(s,t)Q(s)]_{\xi}^{\pi}Q'(t)dt+\int_{\xi}^{\pi}\int_0^{\eta}\Psi(s,t)Q'(s)Q'(t)dsdt$$

(4.12) $$=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

similarly

(4.13) $$I_3=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

Collecting (4.10), (4.11), (4.12) and (4.13), we get

$$V(x,y)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

which proves the theorem A′.

5. Proof theorem D′. Again without the loss of generality we take $C=0$, so that $g(u,v)=\phi(u,v)/16\sin u/2\sin v/2$

we have

$$mn\ B_{mn}=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(s,t)\frac{d}{ds}\left\{\frac{1}{2}+\cos ms\right\}\frac{d}{dt}\left\{\frac{1}{2}+\cos nt\right\}dsdt$$

Thus

$$\bar{S}_{mn}=\sum_1^m\sum_1^n\mu\nu\,B_{\mu\nu}=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(s,t)\frac{d}{ds}\left\{\frac{\sin(m+1/2)s}{2\sin s/2}\right\}\frac{d}{dt}\left\{\frac{\sin(n+1/2)t}{2\sin t/2}\right\}dsdt$$

$$=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \phi(s,t)\left[\frac{2m\cos(m+1/2)s\sin s/2-\sin ms}{4\sin^2 s/2}\right]$$

$$\times\left[\frac{2n\cos(n+1/2)t\cdot\sin t/2-\sin nt}{4\sin^2 t/2}\right]dsdt$$

$$=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi g(s,t)4mn\cos(m+1/2)s\cdot\cos(n+1/2)t\,dsdt$$

$$+\frac{1}{\pi^2}\int_0^\pi\int_0^\pi g(s,t)\frac{\sin ms\sin nt}{\sin s/2\cdot\sin t/2}dsdt$$

$$-\frac{1}{\pi^2}\int_0^\pi\int_0^\pi g(s,t)\frac{2m\cos(m+1/2)s\cdot\sin nt}{\sin t/2}dsdt$$

$$-\frac{1}{\pi^2}\int_0^\pi\int_0^\pi g(s,t)\frac{2n\cos(n+1/2)t\cdot\sin ms}{\sin s/2}dsdt$$

$$=\frac{1}{\pi^2}[I_1+I_2-I_3-I_4]$$

For the proof of the theorem D′, it is required to be shown that

$$\sum_1^\infty\sum_1^\infty \bar{S}_{mn}q^m r^n/mn=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

or equivalently,

(5.1) $$\sum_1^\infty\sum_1^\infty I_i q^m r^n/mn=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

Assume $1-q=\xi,\ 1-r=\eta$

we have $$I_2=\int_0^\pi\int_0^\pi g(s,t)\frac{\sin ms}{s}\,\frac{\sin nt}{t}ds\,dt+o(1)$$

The treatment of this integral is equivalent to establishing the theorem proved in the paper of Singh [4] and accordingly

(5.2) $$\sum_{m=1}^\infty\sum_{n=1}^\infty I_2 q^m r^n/mn=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

As for the integral $I_1$, we have

$$\sum_{m=1}^\infty\sum_{n=1}^\infty I_1 q^m r^n/mn$$

$$=\int_0^\pi\int_0^\pi g(s,t)\,[\lambda(q,s)-\cos s/2]\,[\lambda(r,t)-\cos t/2]\,dsdt$$

$$=\int_0^\pi\int_0^\pi g(s,t)\lambda(q,s)\lambda(r,t)dsdt$$

$$-\int_0^\pi\int_0^\pi g(s,t)\lambda(q,s)\cos t/2\,dsdt$$

$$-\int_0^\pi\int_0^\pi g(s,t)\lambda(r,t)\cos s/2\,dsdt$$

$$+\int_0^\pi\int_0^\pi g(s,t)\cos s/2\cos t/2\,dsdt$$

(5.3) $$=I_{1,1}-I_{1,2}-I_{1,3}+I_{1,4},\ \text{say}$$

Here $$\lambda(r,t)=\lambda(t)=\frac{(1-r)\cos t/2}{1-2r\cos t+r^2}$$

$\lambda(t)$ has the estimates

(5.4) $$\lambda(t)=o\left(\frac{1}{\eta}\right)\qquad t\leq\eta$$

(5.5) $$\lambda(t)=o\left(\frac{1-r}{t^2}\right)=o\left(\frac{1}{t}\right)\quad t>\eta$$

(5.6) $$\lambda'(t)=\frac{d\lambda}{dt}=o\left(\frac{1}{t^2}\right)$$

Also from (3.2)

(5.7) $$st\frac{\partial^2 G}{\partial s\partial t}=g(s,t)\quad \text{almost every where.}$$

Now clearly

(5.8) $$I_{1,4}=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

$$I_{1,1}=\int_0^\pi\int_0^\pi g(s,t)\lambda(s)\lambda(t)dsdt$$

$$=\left(\int_0^\xi\int_0^\eta+\int_\xi^\pi\int_0^\eta+\int_0^\xi\int_\eta^\pi+\int_\xi^\pi\int_\eta^\pi\right)\frac{\partial^2 G}{\partial s\,\partial t}s\lambda(s)\cdot t\lambda(t)dsdt$$

(5.9) $$=I_{1,1,1}+I_{1,1,2}+I_{1,1,3}+I_{1,1,4},\ \text{say.}$$

On integration by parts

$$I_{1,1,1}=[G(\xi,\eta)\lambda(\xi)\lambda(\eta)]-\xi\lambda(\xi)\int_0^\eta G(\xi,t)\frac{d}{dt}[t\lambda(t)]\,dt$$

$$-\eta\lambda(\eta)\int_0^{\xi}G(s,\eta)\frac{d}{ds}[s\lambda(s)]ds+\int_0^{\xi}\int_0^{\eta}G(s,t)\frac{d}{ds}[s\lambda(s)]\frac{d}{dt}[t\lambda(t)]dsdt$$

(5. 10) $$=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

$$I_{1,1,2}=[G(s,t)st\lambda(s)\lambda(t)]_{\xi,0}^{\pi,\eta}-\int_{\xi}^{\pi}[G(s,t)t\lambda(t)]_0^{\eta}\frac{d}{ds}[s\lambda(s)]ds$$

$$-\int_0^{\eta}[G(s,t)s\lambda(s)]_{\xi}^{\pi}\frac{d}{dt}[t\lambda(t)]dt+\int_{\xi}^{\pi}\int_0^{\eta}G(s,t)\frac{d}{ds}[s\lambda(s)]\frac{d}{dt}[t\lambda(t)]dsdt$$

(5. 11) $$=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

Similarly

(5. 12) $$I_{1,1,3}=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

and

$$I_{1,1,4}=[G(s,t)\lambda(s)\lambda(t)st]_{\xi,\eta}^{\pi,\pi}-\int_{\xi}^{\pi}[G(s,t)t\lambda(t)]_{\eta}^{\pi}\frac{d}{ds}[s\lambda(s)]ds$$

$$-\int_{\eta}^{\pi}[G(s,t)s\lambda(s)]_{\xi}^{\pi}\frac{d}{dt}[t\lambda(t)]dt+\int_{\xi}^{\pi}\int_{\eta}^{\pi}G(s,t)\frac{d}{ds}[s\lambda(s)]\frac{d}{dt}[t\lambda(t)]dsdt$$

(5. 13) $$=o\left(\log\frac{1}{\xi}\cdot\log\frac{1}{\eta}\right)=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

combining (5. 10), (5. 11), (5. 12) and (5. 13), we get

$$I_{1,1}=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

Arguing similarly it can be seen that

$$I_{1,2}=I_{1,3}=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

Thus

(5. 14) $$I_1=o\left(\log\frac{1}{1-q}\cdot\log\frac{1}{1-r}\right)$$

Again we consider the typical integral $I_2$, the treatment of $I_3$ will be similar. Here

$$\sum_1^{\infty}\sum_1^{\infty}I_2q^mr^n/mn=\int_0^{\pi}\int_0^{\pi}g(s,t)[\lambda(q,s)-\cos s/2]T(r,t)dsdt$$

where

$$T\equiv T(r,t)=\frac{1}{2}\operatorname{cosec} t/2 \tan^{-1}\frac{r\sin t}{1-r\cos t}$$

The estimate of $T$ are easily seen to be

$$T(r,t)=o(1/\eta) \qquad t\leq\eta$$

$$\frac{dT}{dt}=\begin{cases} o(1/\eta_2) & t\leq\eta \\ o(\eta/t^3) & t>\eta \end{cases}$$

The estimates concerning $T(r,t)$ being exactly similar to that of $\lambda(q,s)$ the integral $I_2$, can be disposed off as before.

Thus

(5.15) $$I_2=o(1)(L),\ I_3=o(1)(L)$$

Collecting (5.1) (5.2) (5.14) (15.15), we prove the theorem D′.

We have considered only the case corresponding to the first allied series. Analogous results can be established for the series corresponding to second and third allied series by proceeding in a similar way.

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