COMPLETIONS OF TOPOLOGICAL VECTOR SPACES

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Let \((X, \mathcal{F})\) be a topological vector space over the real or complex field \(K\). A filter base \(\mathcal{N}\) in \(X\) is said to converge to a point \(a\) in \(X\), written \(\mathcal{N} \to a\), if given any neighborhood \(V\) of the zero vector 0 in \(X\), there exists a set \(N\) in \(\mathcal{N}\) such that \(N \subseteq a + V = \{a + x : x \in V\}\). A filter base \(\mathcal{N}\) in \(X\) is said to be a Cauchy filter base if given any neighborhood \(V\) of the zero vector 0 in \(X\), there exists a set \(N\) in \(\mathcal{N}\) such that \(N - N = \{x - y : x \in N, y \in N\} \subseteq V\). The topological vector space \((X, \mathcal{F})\) is complete if and only if every Cauchy filter base in \(X\) converges to a point in \(X\). It is of course well-known that every Hausdorff topological vector space \((X, \mathcal{F})\) is isomorphic to a dense subspace \((X_\delta, \mathcal{F}|X_\delta)\) of a complete Hausdorff topological vector space \((\hat{X}, \mathcal{F})\); and that the space \((\hat{X}, \mathcal{F})\), called the completion of \((X, \mathcal{F})\), is uniquely determined up to an isomorphism.

The usual constructions of \((\hat{X}, \mathcal{F})\) define \(\hat{X}\) as the set of all equivalence classes of Cauchy filters in \(X\) ([1], pp. 131–134; or [4], pp. 37–49) or as the set of all equivalence classes of Cauchy nets ([2], pp. 33–35 and pp. 148–149) into \(X\). In this note, we give a somewhat simpler construction of the completion \((\hat{X}, \mathcal{F})\) of a Hausdorff topological vector space \((X, \mathcal{F})\) defining \(\hat{X}\) as the set of all equivalence classes of Cauchy filter bases in \(X\).

Using this construction, a comparison of linear spaces obtained as completions of a linear space under different Hausdorff vector topologies is made. Two filter conditions are stated which facilitate such a comparison. One of these is the filter condition introduced by W. Robertson ([3]).

1. Construction of the completion of a Hausdorff topological vector space.

Given a Hausdorff topological vector space \((X, \mathcal{F})\), two filter bases \(\mathcal{N}\) and \(\mathcal{M}\) in \(X\) are said to be equivalent, written \(\mathcal{N} \sim \mathcal{M}\), if given any neighborhood \(V\) of the zero vector 0 in \(X\), there exist sets \(N\) in \(\mathcal{N}\) and \(M\) in \(\mathcal{M}\) such that \(N - M = \{x - y : x \in N \text{ and } y \in M\} \subseteq V\). The relation \(\sim\) is an equivalence relation on the set of all Cauchy filter bases in \(X\); and the set \(\hat{X}\) is defined to be the set of all equivalence classes of Cauchy filter bases in \(X\). The equivalence class-
The sum of two filter bases \( \mathcal{N} \) and \( \mathcal{M} \) in \( X \) is the filter base \( \mathcal{N} + \mathcal{M} = \{ N + M : N \in \mathcal{N} \text{ and } M \in \mathcal{M} \} \) where, as usual, \( N + M = \{ x + y : x \in N \text{ and } y \in M \} \). Clearly, if \( \mathcal{N} \rightarrow \mathbf{z} \) and \( \mathcal{M} \rightarrow \mathbf{b} \), we have \( \mathcal{N} + \mathcal{M} \rightarrow \mathbf{z} + \mathbf{b} \). Also, if \( \mathcal{N} \) and \( \mathcal{M} \) are Cauchy; their sum \( \mathcal{N} + \mathcal{M} \) is Cauchy. This enables us to define addition \( + : \hat{X} \times \hat{X} \rightarrow \hat{X} \) by the correspondence \( [\mathcal{N}] + [\mathcal{M}] = [\mathcal{N} + \mathcal{M}] \) for all \([\mathcal{N}]\) and \([\mathcal{M}]\) in \( \hat{X} \).

The product of a scalar \( \alpha \) in \( K \) and a filter base \( \mathcal{N} \) in \( X \) is the filter base \( \alpha \mathcal{N} = \{ \alpha N : N \in \mathcal{N} \} \) where \( \alpha \mathcal{N} = \{ \alpha x : x \in N \} \). If \( \mathcal{N} \rightarrow \mathbf{z} \), then \( \alpha \mathcal{N} \rightarrow \alpha \mathbf{z} \). If \( \mathcal{N} \) is Cauchy, then \( \alpha \mathcal{N} \) is Cauchy. Thus we define scalar multiplication \( \cdot : K \times \hat{X} \rightarrow \hat{X} \) by the correspondence \( \alpha \cdot [\mathcal{N}] = \alpha [\mathcal{N}] = [\alpha \mathcal{N}] \) for all \( \alpha \) in \( K \) and all \([\mathcal{N}]\) in \( \hat{X} \).

With addition and scalar multiplication so defined, \( \hat{X} \) is a linear space over \( K \). The zero vector in \( \hat{X} \) is the equivalence class \( \{ \{ 0 \} \} \) where 0 is the zero vector in \( X \). If \([\mathcal{N}]\) is an element of \( \hat{X} \), its additive inverse is \(-[\mathcal{N}] = \{-N : N \in \mathcal{N}\}\) and \(-N = \{-x : x \in N\}\).

If \( \mathcal{L} \) is a local base for the vector topology \( \mathcal{T} \), then \( \mathcal{T} \) is the unique vector topology on \( \hat{X} \) determined by the local base \( \mathcal{L} = \{ L : L \in \mathcal{L} \} \) where \( L = \{ [\mathcal{N}] \in \hat{X} : \text{given any neighborhood } V \text{ of } 0 \text{ in } X, \text{ there exists a set } N \in \mathcal{N} \text{ such that } N \subset L + V \} \) for each \( L \) in \( \mathcal{L} \).

With \( \hat{X} \) and \( \mathcal{T} \) thus defined, \( (\hat{X}, \mathcal{T}) \) is a complete Hausdorff topological vector space. The set \( X_d = \{ [\{ x \}] : x \in X \} \) is a dense subset of \( \hat{X} \); and the function \( i : X \rightarrow X_d \), defined by the correspondence \( i(x) = [\{ x \}] \) for all \( x \in X \), is an isomorphism on \( (X, \mathcal{T}) \) onto \( (X_d, \mathcal{T} | X_d) \). The space \( (\hat{X}, \mathcal{T}) \), which is uniquely determined up to an isomorphism, is the completion of \( (X, \mathcal{T}) \).

In the interest of conciseness, proofs of the above results are omitted. They are similar to those given by J. Horváth ([11], pp.131–134) and F. Treves ([4], pp.37–49). The advantages of using equivalence classes of Cauchy filter bases rather than equivalence classes of Cauchy filters are indicated by the above definitions (especially that of addition) and by the following remarks:

1. If \( a \in X \): then \( \{ a \} \) is a filter base in \( X \) which converges to \( a \) and \( i(a) = [\{ a \}] \) is the equivalence class (element of \( \hat{X} \)) consisting of all the Cauchy filter bases in \( X \) which converge to \( a \).
(2) If $\mathcal{N}$ is a Cauchy filter base in $X$, i.e., if $[\mathcal{N}] \in \hat{X}$: then $i(\mathcal{N}) = \{i(N) : N \in \mathcal{N}\}$, where $i(N) = \{i(x) : x \in N\}$, is a Cauchy filter base in $X_d$ such that $i(\mathcal{N}) \rightarrow \mu[\mathcal{N}]$.

(3) If $A \subset X$ and $[\mathcal{N}] \in \hat{X}$: then $[\mathcal{N}] \in \bar{i}(A)$ if and only if there exists a Cauchy filter base $\mathcal{M}$ in $A$ such that $\mathcal{M} \rightarrow \mu[\mathcal{N}]$.

Incidently, if $p : X \rightarrow R$ is a continuous semi-norm on $X$ then $p \circ i^{-1} : X_d \rightarrow R$ is a continuous semi-norm on $X_d$ which has a unique (uniformly) continuous extension $\hat{p} : \hat{X} \rightarrow R$ to $\hat{X}$. Using our construction of $\hat{X}$, we see that $\hat{p} : \hat{X} \rightarrow R$ is given by $\hat{p}([\mathcal{N}]) = \lim_{x \rightarrow \mathcal{N}} p(x)$ for all $[\mathcal{N}] \in \hat{X}$. If $(X, \mathcal{F})$ is a Hausdorff locally convex topological vector space with vector topology $\mathcal{F}$ generated by the non-empty family of semi-norms $\mathcal{P}$; then $(\hat{X}, \hat{\mathcal{F}})$ is a Hausdorff locally convex topological vector space with vector topology $\hat{\mathcal{F}}$ generated by the non-empty family of semi-norms $\{\hat{p} : \mathcal{P} \in \mathcal{F}\}$.

If $f : X \rightarrow X$ is uniformly continuous (or, more generally, Cauchy-regular in the sense that $f$ preserves Cauchy filter bases): then the function $f_d : X_d \rightarrow X_d$, defined by $f_d = i \circ f \circ i^{-1}$ is uniformly continuous (or Cauchy-regular) and has a unique uniformly continuous (or Cauchy-regular) extension by continuity to a function $\hat{f} : \hat{X} \rightarrow \hat{X}$. If $f$ is linear, then $\hat{f}$ is linear. Using our construction, $\hat{f} : \hat{X} \rightarrow \hat{X}$ is given by $\hat{f}([\mathcal{N}]) = [f(\mathcal{N})]$ for all $[\mathcal{N}] \in \hat{X}$.

2. The filter conditions.

Let $X$ be a linear space over the real or complex field $K$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Hausdorff vector topologies on $X$ such that $\mathcal{F}_1 \subset \mathcal{F}_2$. Let $(\hat{X}_1, \hat{\mathcal{F}}_1)$ be the completion of $(X, \mathcal{F}_1)$ with $(X, \mathcal{F}_2)$ being isomorphic to the dense subspace $(X_d, \mathcal{F}_1 | X_d)$ of $(\hat{X}_1, \hat{\mathcal{F}}_1)$ under the isomorphism $i_1 : X \rightarrow X_d$. Let $(\hat{X}_2, \hat{\mathcal{F}}_2)$ be the completion of $(X, \mathcal{F}_2)$ with $(X, \mathcal{F}_2)$ being isomorphic to the dense subspace $(X_d, \mathcal{F}_2 | X_d)$ of $(\hat{X}_2, \hat{\mathcal{F}}_2)$ under the isomorphism $i_2 : X \rightarrow X_d$. The function $i : X_d \rightarrow X_d$, defined by $i = i_1 \circ i_2^{-1}$, is a $\hat{\mathcal{F}}_2$, $\hat{\mathcal{F}}_1$-continuous algebraic isomorphism on $X_d$ onto $X_d$. Let $\hat{i} : \hat{X}_2 \rightarrow \hat{X}_1$ be the unique $\hat{\mathcal{F}}_2$, $\hat{\mathcal{F}}_1$-continuous extension of $i$ to $\hat{X}_2$. Clearly, $\hat{i}$ is linear and uniformly continuous.

THEOREM 1. The function $\hat{i}$ is $1-1$ if and only if the following filter condition (see [3], p.244) holds:

(A) If $\mathcal{N}$ is a $\mathcal{F}_2$-Cauchy filter base in $X$ and if $\mathcal{N}$ is $\mathcal{F}_1$-convergent to 0, then $\mathcal{N}$ is $\mathcal{F}_2$-convergent to 0.
PROOF. Assume that the filter condition (A) holds. We must show that \( \hat{i} \) is 1-1, or equivalently, that \( \ker \hat{i}^{-1}([\{0\}]_1) = [\{0\}]_2 \). Let \([\mathcal{M}]_2 \in \ker \hat{i} \) i.e., let \([\mathcal{M}]_2 \in \hat{X}_2 \) such that \( \hat{i}([\mathcal{M}]_2) = [\{0\}]_2 \). By (2), we see that \( \mathcal{M} \) is a \( \mathcal{F}_2 \)-Cauchy filter base in \( X \) and \( i_2(\mathcal{M}) \) is a \( \mathcal{F}_2 \)-Cauchy filter base in \( X_d \), such that \( i_2(\mathcal{M}) \) is \( \mathcal{F}_1 \)-convergent to \([\mathcal{M}]_2 \). Since \( \hat{i} \) is continuous, \( \hat{i}(i_2(\mathcal{M})) \) is \( \mathcal{F}_1 \)-convergent to \( \hat{i}([\mathcal{M}]_2) \). But \( \hat{i}(i_2(\mathcal{M})) = i_1(\mathcal{M}) \) and \( \hat{i}([\mathcal{M}]_2) = [\{0\}]_1 \). Thus \( i_1(\mathcal{M}) \) is \( \mathcal{F}_1 \)-convergent to \([\{0\}]_1 \). Since \( i_1 \) is continuous, \( i_1(\mathcal{M}) \) is \( \mathcal{F}_1 \)-convergent to \([\{0\}]_2 \). Since \( i_1^{-1} \) is continuous, \( \mathcal{M} \) is \( \mathcal{F}_1 \)-convergent to \( 0 \). By the filter condition (A), we see that \( \hat{i} \) is \( 1 \)-1. Conversely, assume that \( \hat{i} \) is 1-1. We must show that the filter condition (A) holds. Let \([\mathcal{M}]_2 \in \ker \hat{i} \), i.e., let \([\mathcal{M}]_2 \in \hat{X}_2 \) such that \( \hat{i}(\mathcal{M}) = [\{0\}]_1 \). By (2), we see that \( \mathcal{M} \) is a \( \mathcal{F}_2 \)-Cauchy filter base in \( X \) and \( \mathcal{F}_1 \)-convergent to \( [\mathcal{M}]_2 \). Since \( \hat{i} \) is continuous, \( \hat{i}(i_2(\mathcal{M})) \) is \( \mathcal{F}_1 \)-convergent to \( \hat{i}([\mathcal{M}]_2) \). But \( \hat{i}(i_2(\mathcal{M})) = i_1(\mathcal{M}) \) and \( \hat{i}([\mathcal{M}]_2) = [\{0\}]_1 \). Thus \( i_1(\mathcal{M}) \) is \( \mathcal{F}_1 \)-convergent to \([\{0\}]_2 \). Since \( i_1^{-1} \) is continuous, \( \mathcal{M} \) is \( \mathcal{F}_1 \)-convergent to \( 0 \). By the filter condition (A), we see that \( \mathcal{M} \) is \( \mathcal{F}_2 \)-convergent to \( 0 \). Thus, by (1), we have \([\mathcal{M}]_2 = [\{0\}]_2 \). Thus \( \hat{i} \) is 1-1.

Conversely, assume that \( \hat{i} \) is onto \( \hat{X}_1 \). We must show that the filter condition (A) holds. Let \([\mathcal{M}]_2 \in \ker \hat{i} \), then \( \mathcal{M} \) is a \( \mathcal{F}_2 \)-Cauchy filter base in \( X \) such that \( \mathcal{M} \) is \( \mathcal{F}_1 \)-equivalent to \( \mathcal{M} \). Clearly, \([\mathcal{M}]_2 = [\mathcal{M}]_1 \). Since \( \mathcal{M} \) is a \( \mathcal{F}_2 \)-Cauchy filter base in \( X \), the set \( i_2(\mathcal{M}) \) is a \( \mathcal{F}_2 \)-Cauchy filter base in \( X_d \), which is \( \mathcal{F}_2 \)-convergent to \([\mathcal{M}]_2 \). Also, since \( \hat{i} \) is continuous, \( \hat{i}(i_2(\mathcal{M})) \) is \( \mathcal{F}_1 \)-convergent to \( \hat{i}([\mathcal{M}]_2) \) whence \( i_1(\mathcal{M}) \) is \( \mathcal{F}_1 \)-convergent to \( \hat{i}([\mathcal{M}]_2) \). But \( \mathcal{M} \) is a \( \mathcal{F}_1 \)-Cauchy filter base in \( X \); so \( i_1(\mathcal{M}) \) is a \( \mathcal{F}_1 \)-Cauchy filter base in \( X_d \), which is \( \mathcal{F}_1 \)-convergent to \([\mathcal{M}]_2 \). Since \( \hat{X}_1 \), \( \mathcal{F}_1 \) is Hausdorff, we have \( \hat{i}([\mathcal{M}]_2) = [\mathcal{M}]_1 = [\mathcal{M}]_2 \). This proves that \( \hat{i} \) is onto \( \hat{X}_1 \).

Conversely, assume that \( \hat{i} \) is onto \( \hat{X}_1 \). We must show that filter condition (B)
holds. Let $\mathcal{N}$ be a $\mathcal{F}_1$-Cauchy filter base in $X$. Then $[\mathcal{N}]_1 \subseteq X_1$. Since $i$ is onto $X_1$, there exists a point $[\mathcal{M}]_2$ in $X_2$ such that $i([\mathcal{M}]_2) = [\mathcal{N}]_1$. Clearly, $\mathcal{M}$ is a $\mathcal{F}_2$-Cauchy filter base in $X$. We must show that $\mathcal{M}$ is $\mathcal{F}_1$-equivalent to $\mathcal{N}$. Since $\mathcal{M}$ is a $\mathcal{F}_2$-Cauchy filter base in $X$, the set $i_2(\mathcal{M})$ is a $\mathcal{F}_2$-Cauchy filter base in $X_d$, which is $\mathcal{F}_1$-convergent to $[\mathcal{M}]_2$. Since $i$ is continuous, since $i(i_2(\mathcal{M})) = i_1(\mathcal{M})$, and since $i([\mathcal{M}]_2) = [\mathcal{N}]_1$; the filter base $i_1(\mathcal{M})$ is $\mathcal{F}_1$-convergent to $[\mathcal{N}]_1$. Also, since $\mathcal{N}$ is a $\mathcal{F}_1$-Cauchy filter base in $X$: the filter base $i_1(\mathcal{N})$ in $X_d$ is $\mathcal{F}_1$-convergent to $[\mathcal{N}]_1$. Thus $i_1(\mathcal{M})$ and $i_1(\mathcal{N})$ are $\mathcal{F}_1$-equivalent. Since $i_1^{-1}$ is uniformly continuous, $\mathcal{M}$ is $\mathcal{F}_1$-equivalent to $\mathcal{N}$. This proves that filter condition (B) holds.

If filter condition (A) holds, the linear space $X_2$ is algebraically isomorphic to a linear subspace of the linear space $X_1$; and of course the set $X_2$ is equipotent to a subset of the set $X_1$, written $X_2 \subseteq X_1$. If filter conditions (A) and (B) hold, the linear space $X_2$ is algebraically isomorphic to the linear space $X_1$; and the set $X_2$ is equipotent to the set $X_1$, written $X_2 \sim X_1$. Finally, if filter condition (A) holds and filter condition (B) does not hold, the linear space $X_2$ is algebraically isomorphic to a proper linear subspace of the linear space $X_1$; and the set $X_2$ is equipotent to a proper subset of the set $X_1$, written $X_2 \subsetneq X_1$.

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