Kyungpook Math. J.
Volume 14, Number 1
June, 1974

COMPLETIONS OF TOPOLOGICAL VECTOR SPACES

By Ray F. Snipes

Let $(X, \mathscr{T})$ be a topological vector space over the real or complex field K. A filter base $\mathscr{N}$ in X is said to *converge* to a point a in X, written $\mathscr{N} \longrightarrow a$, if given any neighborhood V of the zero vector 0 in X, there exists a set N in $\mathscr{N}$ such that $N \subset a+V=\{a+x : x \in V\}$. A filter base $\mathscr{N}$ in X is said to be a *Cauchy filter base* if given any neighborhood V of the zero vector 0 in X, there exists a set N in $\mathscr{N}$ such that $N-N=\{x-y : x, y \in N\} \subset V$. The topological vector space $(X, \mathscr{T})$ is *complete* if and only if every Cauchy filter base in X converges to a point in X. It is of course well-known that every Hausdorff topological vector space $(X, \mathscr{T})$ is isomorphic to a dense subspace $(X_d, \hat{\mathscr{T}} | X_d)$ of a complete Hausdorff topological vector space $(\hat{X}, \hat{\mathscr{T}})$; and that the space $(\hat{X}, \hat{\mathscr{T}})$, called the *completion* of $(X, \mathscr{T})$, is uniquely determined up to an isomorphism.

The usual constructions of $(\hat{X}, \hat{\mathscr{T}})$ define $\hat{X}$ as the set of all equivalence classes of Cauchy filters in X ([1], pp. 131—134; or [4], pp. 37—49) or as the set of all equivalence classes of Cauchy nets ([2], pp. 33—35 and pp. 148—149) into X. In this note, we give a somewhat simplier construction of the completion $(\hat{X}, \hat{\mathscr{T}})$ of a Hausdorff topological vector space $(X, \mathscr{T})$ defining $\hat{X}$ as the set of all equivalence classes of Cauchy filter bases in X.

Using this construction, a comparison of linear spaces obtained as completions of a linear space under different Hausdorff vector topologies is made. Two filter conditions are stated which facilitate such a comparison. One of these is the filter condition introduced by W. Robertson ([3]).

1. Construction of the completion of a Hausdorff topological vector space.

Given a Hausdorff topological vector space $(X, \mathscr{T})$, two filter bases $\mathscr{N}$ and $\mathscr{M}$ in X are said to be *equivalent*, written $\mathscr{N} \overset{\mathscr{T}}{\sim} \mathscr{M}$, if given any neighborhood V of the zero vector 0 in X, there exist sets N in $\mathscr{N}$ and M in $\mathscr{M}$ such that $N-M=\{x-y : x \in N \text{ and } y \in M\} \subset V$. The relation $\sim$ is an equivalence relation on the set of all Cauchy filter bases in X; and the set $\hat{X}$ is defined to be the set of all equivalence classes of Cauchy filter bases in X. The equivalence class (element of $\hat{X}$) with representative the Cauchy filter base $\mathscr{N}$ is denoted by $[\mathscr{N}]$. Thus we write $[\mathscr{N}]=\{\mathscr{M} : \mathscr{M}$ is a Cauchy filter base in X and $\mathscr{M}\overset{\mathscr{T}}{\sim}\mathscr{N}\}$.

The sum of two filter bases $\mathscr{N}$ and $\mathscr{M}$ in X is the filter base $\mathscr{N}+\mathscr{M}=\{N+M : N\in\mathscr{N}$ and $M\in\mathscr{M}\}$ where, as usual, $N+M=\{x+y : x\in N$ and $y\in M\}$. Clearly, if $\mathscr{N}\xrightarrow{\mathscr{T}}a$ and $\mathscr{M}\xrightarrow{\mathscr{T}}b$, we have $\mathscr{N}+\mathscr{M}\xrightarrow{\mathscr{T}}a+b$. Also, if $\mathscr{N}$ and $\mathscr{M}$ are Cauchy; their sum $\mathscr{N}+\mathscr{M}$ is Cauchy. This enables us to define addition $+ : \hat{X}\times\hat{X}\longrightarrow\hat{X}$ by the correspondence $[\mathscr{N}]+[\mathscr{M}]=[\mathscr{N}+\mathscr{M}]$ for all $[\mathscr{N}]$ and $[\mathscr{M}]$ in $\hat{X}$.

The product of a scalar α in K and a filter base $\mathscr{N}$ in X is the filter base $\alpha\mathscr{N}=\{\alpha N : N\in\mathscr{N}\}$ where $\alpha N=\{\alpha x : x\in N\}$. If $\mathscr{N}\xrightarrow{\mathscr{T}}a$, then $\alpha\mathscr{N}\xrightarrow{\mathscr{T}}\alpha a$. If $\mathscr{N}$ is Cauchy, then $\alpha\mathscr{N}$ is Cauchy. Thus we define scalar multiplication $\cdot : K\times\hat{X}\longrightarrow\hat{X}$ by the correspondence $\alpha\cdot[\mathscr{N}]=\alpha[\mathscr{N}]=[\alpha\mathscr{N}]$ for all α in K and all $[\mathscr{N}]$ in $\hat{X}$.

With addition and scalar multiplication so defined, $\hat{X}$ is a linear space over K. The zero vector in $\hat{X}$ is the equivalence class $[\{\{0\}\}]$ where 0 is the zero vector in X. If $[\mathscr{N}]$ is an element of $\hat{X}$, its additive inverse is $-[\mathscr{N}]=[-\mathscr{N}]$ where $-\mathscr{N}=\{-N : N\in\mathscr{N}\}$ and $-N=\{-x : x\in N\}$.

If $\mathscr{L}$ is a local base for the vector topology $\mathscr{T}$, then $\hat{\mathscr{T}}$ is the unique vector topology on $\hat{X}$ determined by the local base $\hat{\mathscr{L}}=\{\hat{L} : L\in\mathscr{L}\}$ where $\hat{L}=\{[\mathscr{N}]\in\hat{X} :$ given any neighborhood V of 0 in X, there exists a set $N\in\mathscr{N}$ such that $N\subset L+V\}$ for each L in $\mathscr{L}$.

With $\hat{X}$ and $\hat{\mathscr{T}}$ thus defined, $(\hat{X}, \hat{\mathscr{T}})$ is a complete Hausdorff topological vector space. The set $X_d=\{[\{\{x\}\}] : x\in X\}$ is a dense subset of $\hat{X}$; and the function i: $X\longrightarrow X_d$, defined by the correspondence $i(x)=[\{\{x\}\}]$ for all $x\in X$, is an isomorphism on $(X, \mathscr{T})$ onto $(X_d, \hat{\mathscr{T}}|X_d)$. The space $(\hat{X}, \hat{\mathscr{T}})$, which is uniquely determined up to an isomorphism, is the completion of $(X, \mathscr{T})$.

In the interest of conciseness, proofs of the above results are omitted. They are similar to those given by J. Horváth ([1], pp. 131—134) and F. Treves ([4], pp. 37—49). The advantages of using equivalence classes of Cauchy filter bases rather than equivalence classes of Cauchy filters are indicated by the above definitions (especially that of addition) and by the following remarks:

(1) If $a\in X$: then $\{\{a\}\}$ is a filter base in X which converges to a and $i(a)=[\{\{a\}\}]$ is the equivalence class (element of $\hat{X}$) consisting of all the Cauchy filter bases in X which converge to a.

(2) If $\mathscr{N}$ is a Cauchy filter base in X, i.e., if $[\mathscr{N}]\in\hat{X}$; then $i(\mathscr{N})=\{i(N): N\in\mathscr{N}\}$, where $i(N)=\{i(x): x\in N\}$, is a Cauchy filter base in X_d such that $i(\mathscr{N})\xrightarrow{\hat{\mathscr{T}}}[\mathscr{N}]$.

(3) If $A\subset X$ and $[\mathscr{N}]\in\hat{X}$: then $[\mathscr{N}]\in\overline{i(A)}$ if and only if there exists a Cauchy filter base $\mathscr{M}$ in A such that $\mathscr{M}\overset{\mathscr{T}}{\sim}\mathscr{N}$ or $[\mathscr{M}]=[\mathscr{N}]$.

Incidently, if $p: X\longrightarrow R$ is a continuous semi-norm on X then $p\circ i^{-1}: X_d\longrightarrow R$ is a continuous semi-norm on X_d which has a unique (uniformly) continuous extension $\hat{p}: \hat{X}\longrightarrow R$ to $\hat{X}$. Using our construction of $\hat{X}$, we see that $\hat{p}: \hat{X}\longrightarrow R$ is given by $p([\mathscr{N}])=\lim_{x,\mathscr{N}} p(x)$ for all $[\mathscr{N}]\in\hat{X}$. If $(X,\mathscr{T})$ is a Hausdorff locally convex topological vector space with vector topology $\mathscr{T}$ generated by the non-empty family of semi-norms $\mathscr{P}$: then $(\hat{X},\hat{\mathscr{T}})$ is a Hausdorff locally convex topological vector space with vector topology $\hat{\mathscr{T}}$ generated by the non-empty family of semi-norms $\{\hat{P}: P\in\mathscr{P}\}$.

If $f: X\longrightarrow X$ is uniformly continuous (or, more generally, Cauchy-regular in the sense that f preserves Cauchy filter bases): then the function $f_d: X_d\longrightarrow X_d$ defined by $f_d=i\circ f\circ i^{-1}$ is uniformly continuous (or Cauchy-regular) and has a unique uniformly continuous (or Cauchy-regular) extension by continuity to a function $\hat{f}: \hat{X}\longrightarrow\hat{X}$. If f is linear, then $\hat{f}$ is linear. Using our construction, $\hat{f}: \hat{X}\longrightarrow\hat{X}$ is given by $\hat{f}([\mathscr{N}])=[f(\mathscr{N})]$ for all $[\mathscr{N}]\in\hat{X}$.

2. The filter conditions.

Let X be a linear space over the real or complex field K. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two Hausdorff vector topologies on X such that $\mathscr{T}_1\subset\mathscr{T}_2$. Let $(\hat{X}_1,\hat{\mathscr{T}}_1)$ be the completion of $(X,\mathscr{T}_1)$ with $(X,\mathscr{T}_1)$ being isomorphic to the dense subspace $(X_{d_1}, \hat{\mathscr{T}}_1|X_{d_1})$ of $(\hat{X}_1, \hat{\mathscr{T}}_1)$ under the isomorphism $i_1: X\longrightarrow X_{d_1}$. Let $(\hat{X}_2,\hat{\mathscr{T}}_2)$ be the completion of $(X, \mathscr{T}_2)$ with $(X, \mathscr{T}_2)$ being isomorphic to the dense subspace $(X_{d_2}, \hat{\mathscr{T}}_2|X_{d_2})$ of $(\hat{X}_2,\hat{\mathscr{T}}_2)$ under the isomorphism $i_2: X\longrightarrow X_{d_2}$. The function $i: X_{d_2}\longrightarrow X_{d_1}$, defined by $i=i_1\circ i_2^{-1}$, is a $\hat{\mathscr{T}}_2$, $\hat{\mathscr{T}}_1$-continuous algebraic isomorphism on X_{d_2} onto X_{d_1}. Let $\hat{i}: \hat{X}_2\longrightarrow\hat{X}_1$ be the unique $\hat{\mathscr{T}}_2$, $\hat{\mathscr{T}}_1$-continuous extension of i to $\hat{X}_2$. Clearly, $\hat{i}$ is linear and uniformly continuous.

THEOREM 1. *The function $\hat{i}$ is 1—1 if and only if the following filter condition* (see [3], p. 244) *holds*:

(A) *If $\mathscr{N}$ is a $\mathscr{T}_2$-Cauchy filter base in X and if $\mathscr{N}$ is $\mathscr{T}_1$-convergent to* 0, *then $\mathscr{N}$ is $\mathscr{T}_2$-convergent to* 0.

PROOF. Assume that the filter condition (A) holds. We must show that $\hat{i}$ is 1—1, or equivalently, that ker $\hat{i}=\hat{i}^{-1}(\{[\{\{0\}\}]_1\})=\{[\{\{0\}\}]_2\}$. Let $[\mathscr{N}]_2\in$ ker $\hat{i}$, i.e., let $[\mathscr{N}]_2\in\hat{X}_2$ such that $\hat{i}([\mathscr{N}]_2)=[\{\{0\}\}]_1$. By (2), we see that $\mathscr{N}$ is a $\mathscr{T}_2$-Cauchy filter base in X and $i_2(\mathscr{N})$ is a $\hat{\mathscr{T}}_2$-Cauchy filter base in X_{d_2} such that $i_2(\mathscr{N})$ is $\hat{\mathscr{T}}_2$-convergent to $[\mathscr{N}]_2$. Since $\hat{i}$ is continuous, $\hat{i}(i_2(\mathscr{N}))$ is $\hat{\mathscr{T}}_1$-convergent to $\hat{i}([\mathscr{N}]_2)$. But $\hat{i}(i_2(\mathscr{N}))=i_1(\mathscr{N})$; and $\hat{i}([\mathscr{N}]_2)=[\{\{0\}\}]_1$. Thus $i_1(\mathscr{N})$ is $\hat{\mathscr{T}}_1$-convergent to $[\{\{0\}\}]_1$. Since i_1^{-1} is continuous, $\mathscr{N}$ is $\mathscr{T}_1$-convergent to 0. By the filter condition (A), we see that $\mathscr{N}$ is $\mathscr{T}_2$-convergent to 0. Thus, by (1), we have $[\mathscr{N}]_2=[\{\{0\}\}]_2$. Thus $\hat{i}$ is 1—1.

Conversely, assume that $\hat{i}$ is 1—1. We must show that the filter condition (A) holds. Let $\mathscr{N}$ be a $\mathscr{T}_2$-Cauchy filter base in X such that $\mathscr{N}$ is $\mathscr{T}_1$-convergent to 0. Then $i_2(\mathscr{N})$ is a $\hat{\mathscr{T}}_2$-Cauchy filter base in X_{d_2} which is $\hat{\mathscr{T}}_2$-convergent to $[\mathscr{N}]_2$ where $[\mathscr{N}]_2\in\hat{X}_2$. Since $\hat{i}$ is continuous, $\hat{i}(i_2(\mathscr{N}))$ is $\hat{\mathscr{T}}_1$-convergent to $\hat{i}([\mathscr{N}]_2)$. Thus $i_1(\mathscr{N})$ is $\hat{\mathscr{T}}_1$-convergent to $\hat{i}([\mathscr{N}]_2)$. Since i_1 is continuous, $i_1(\mathscr{N})$ is also $\hat{\mathscr{T}}_1$-convergent to $[\{\{0\}\}]_1$. Since $(\hat{X}_1, \hat{\mathscr{T}}_1)$ is Hausdorff, $\hat{i}([\mathscr{N}]_2)=[\{\{0\}\}]_1$ or $[\mathscr{N}]_2\in$ ker $\hat{i}$ whence $[\mathscr{N}]_2=[\{\{0\}\}]_2$. Thus $\mathscr{N}$ is $\mathscr{T}_2$-convergent to 0. This proves that the filter condition (A) holds.

THEOREM 2. *The function $\hat{i}$ is onto $\hat{X}_1$ if and only if the following filter condition holds*:

(B) *Given any $\mathscr{T}_1$-Cauchy filter base $\mathscr{N}$ in X, there exists a $\mathscr{T}_2$-Cauchy filter base $\mathscr{M}$ in X such that $\mathscr{M}$ is $\mathscr{T}_1$-equivalent to $\mathscr{N}$.*

PROOF. Assume that the filter condition (B) holds. We must show that $\hat{i}$ is onto $\hat{X}_1$. Let $[\mathscr{N}]_1\in\hat{X}_1$. Then $\mathscr{N}$ is a $\mathscr{T}_1$-Cauchy filter base in X. By the filter condition (B), there exists a $\mathscr{T}_2$-Cauchy filter base $\mathscr{M}$ in X such that $\mathscr{M}$ is $\mathscr{T}_1$-equivalent to $\mathscr{N}$. We will show that $\hat{i}([\mathscr{M}]_2)=[\mathscr{N}]_1$. Clearly, $[\mathscr{N}]_1=[\mathscr{M}]_1$. Since $\mathscr{M}$ is a $\mathscr{T}_2$-Cauchy filter base in X, the set $i_2(\mathscr{M})$ is a $\hat{\mathscr{T}}_2$-Cauchy filter base in X_{d_2} which is $\hat{\mathscr{T}}_2$-convergent to $[\mathscr{M}]_2$. Also, since $\hat{i}$ is continuous, $\hat{i}(i_2(\mathscr{M}))$ is $\hat{\mathscr{T}}_1$ -convergent to $\hat{i}([\mathscr{M}]_2)$ whence $i_1(\mathscr{M})$ is $\hat{\mathscr{T}}_1$-convergent to $\hat{i}([\mathscr{M}]_2)$. But $\mathscr{M}$ is a $\mathscr{T}_1$-Cauchy filter base in X; so $i_1(\mathscr{M})$ is a $\hat{\mathscr{T}}_1$-Cauchy filter base in X_{d_1} which is $\hat{\mathscr{T}}_1$-convergent to $[\mathscr{M}]_1$. Since $(\hat{X}_1, \hat{\mathscr{T}}_1)$ is Hausdorff, we have $\hat{i}([\mathscr{M}]_2)=[\mathscr{M}]_1=[\mathscr{N}]_1$. This proves that $\hat{i}$ is onto $\hat{X}_1$.

Conversely, assume that $\hat{i}$ is onto $\hat{X}_1$. We must show that filter condition (B) holds. Let $\mathscr{N}$ be a $\mathscr{T}_1$-Cauchy filter base in X. Then $[\mathscr{N}]_1 \in \hat{X}_1$. Since $\hat{i}$ is onto $\hat{X}_1$, there exists a point $[\mathscr{M}]_2$ in $\hat{X}_2$ such that $\hat{i}([\mathscr{M}]_2)=[\mathscr{N}]_1$. Clearly, $\mathscr{M}$ is a $\mathscr{T}_2$-Cauchy filter base in X. We must show that $\mathscr{M}$ is $\mathscr{T}_1$-equivalent to $\mathscr{N}$. Since $\mathscr{M}$ is a $\mathscr{T}_2$-Cauchy filter base in X, the set $i_2(\mathscr{M})$ is a $\hat{\mathscr{T}}_2$-Cauchy filter base in X_{d_2} which is $\hat{\mathscr{T}}_2$-convergent to $[\mathscr{M}]_2$. Since $\hat{i}$ is continuous, since $\hat{i}(i_2(\mathscr{M}))=i_1(\mathscr{M})$, and since $\hat{i}([\mathscr{M}]_2)=[\mathscr{N}]_1$; the filter base $i_1(\mathscr{M})$ is $\hat{\mathscr{T}}_1$-convergent to $[\mathscr{N}]_1$. Also, since $\mathscr{N}$ is a $\mathscr{T}_1$-Cauchy filter base in X ; the filter base $i_1(\mathscr{N})$ in X_{d_1} is $\hat{\mathscr{T}}_1$-convergent to $[\mathscr{N}]_1$. Thus $i_1(\mathscr{M})$ and $i_1(\mathscr{N})$ are $\hat{\mathscr{T}}_1$-equivalent. Since i_1^{-1} is uniformly continuous, $\mathscr{M}$ is $\mathscr{T}_1$-equivalent to $\mathscr{N}$. This proves that filter condition (B) holds.

If filter condition (A) holds, the linear space $\hat{X}_2$ is algebraically isomorphic to a linear subspace of the linear space $\hat{X}_1$: and of course the set $\hat{X}_2$ is equipotent to a subset of the set $\hat{X}_1$, written $\hat{X}_2 \lesssim \hat{X}_1$. If filter conditions (A) and (B) hold, the linear space $\hat{X}_2$ is algebraically isomorphic to the linear space $\hat{X}_1$; and the set $\hat{X}_2$ is equipotent to the set $\hat{X}_1$, written $\hat{X}_2 \sim \hat{X}_1$. Finally, if filter condition (A) holds and filter condition (B) does not hold, the linear space $\hat{X}_2$ is algebraically isomorphic to a proper linear subspace of the linear space $\hat{X}_1$; and the set $\hat{X}_2$ is equipotent to a proper subset of the set $\hat{X}_1$, written $\hat{X}_2 < \hat{X}_1$.

Bowling Green State University
Bowling Green, Ohio

REFERENCES

[1] J. Horváth, *Topological Vector Spaces and Distributions.* Addison-Wesley, Reading, Mass., 1966.

[2] G. Köthe, *Topological Vector Spaces I.* Springer-Verlag, New York, 1969.

[3] W. Robertson, *Completions of Topological Vector Spaces.* Proc. London Math. Soc. (3) 8 (1958), pp. 242—257.

[4] F. Treves, *Topological Vector Spaces, Distributions and Kernels.* Academic Press, New York, 1967.