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CLASSICAL RINGS OF QUOTIENTS OF GROUP-RINGS

By Keng-Teh Tan

1. Introduction.

In this paper, our main purpose is to give a different approach to the following theorem:

If R is an order in an (left) artinian ring and G a poly-(cyclic or finite) group, then the group-ring RG is again an order in an artinian ring.

This result was first proved by P.F. Smith [9]. A slightly more special case of this theorem in which R is semiprime left Goldie was obtained independently by the author in his doctoral dissertation submitted to Queen's University at Kingston, Canada [12]. However, in our approach given here, we first develop and study the concepts of group-rings with factor sets and prove a few results including the fundamental theorem of twisted group-rings. With this machinery and Small's characterization of noetherian rings which are orders in artinian rings, we obtain an alternative proof of the theorem. Perhaps the machinery developed in this paper will be of some use in the study of group-rings of certain group extensions.

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2. Preliminaries.

All rings considered are associative and possess identities. Operations of groups will be denoted multiplicatively. I will denote both ring identities as well as group identities. All terminologies are standard. However, some of the key terminologies are given below.

A group G is called *poly-(cyclic or finite)* if it has a subnormal series such that each factor is either cyclic or finite. G is called *poly-(infinite cyclic)* if it has a subnormal series in which every factor is infinite cyclic.

A theorem of K.A. Hirsch ([3], §2.1) says that if G is poly-(cyclic or finite), then G has a normal subgroup H of finite index in G such that H is poly-(infinite cyclic).

A ring S is a *partial left quotient ring of* R if (1) R is a subring of S and (2) $s \in S$ implies $s=r^{-1}a$ for some $r, a \in R$ with r regular in R.

Let R be a ring and M a multiplicatively closed subset of regular elements of R. Then R is said to satisfy the *left Ore condition with respect to* M if for each $m \in M$, $r \in R$, there exist $m' \in M$, $r' \in R$ such that $m'r=r'm$. In this case, we call M an *Ore semi-group of* R. If M is one such, then there exists a ring R_M such that (1) $R \subset R_M$, (2) $m \in M$ implies that m is a unit in R_M, (3) $x \in R_M$ implies that $x=m^{-1}a$ for some $m \in M$, $a \in R$. R_M is unique up to isomorphism. If M happens to be the set of all regular elements of R, then R_M is called the *total left quotient ring of* R and it is usually denoted by $Q(R)$. It is well known that R has a total left quotient ring if and only if the set of all its regular elements forms an Ore semi-group of R ([4]). A ring R is called an *Ore ring* if $Q(R)$ exists, and in this case, R is said to be an *order in* $Q(R)$.

We now quote some results that will be used later.

(2.1) *If* S *is a partial quotient ring of a ring* R, *then* $S=R_M$ *for some Ore semi-group* M *of* R.

(2.2) *If* R *is semi-prime and* $Q(R)$ *exists, then* $Q(R)$ *is also semi-prime.*

(2.3) *If* R *is (left) artinian, then* $Q(R)=R$ ([7], 2.3).

(2.4) Goldie's Theorem: *R is semi-prime left Goldie if and only if $Q(R)$ exists and is semi-simple (trivial Jacobson radical) artinian* ([2], pp. 166~169).

Let M be an Ore semi-group of R and $S=R_M$.

(2.5) The Common Denominator Theorem: *For* $s_1, \cdots, s_n \in S$, *there exist* $m \in M$, $r_1, \cdots, r_n \in R$ *such that* $s_i=m^{-1}r_i$ *for all* $i=1, \cdots, n$ ([4], p. 6).

(2.6) *If* $Q(S)$ *exists and is artinian, then* $Q(R)$ *exists and in fact* $Q(R)=Q(S)$ ([8], p. 16).

(2.7) *If* R *is (left) noetherian, then so is* S ([8], p. 16).

(2.8) *Let* $R=R_1 \oplus \cdots \oplus R_n$. *Then* $Q(R)$ *exists if and only if each* $Q(R_i)$ *exists and in this case* $Q(R)=Q(R_1) \oplus \cdots \oplus Q(R_n)$ ([8], p. 29).

(2.9) *If* $S=Q(R)$ *is artinian, then* $Q(R_{n\times n})=S_{n\times n}$, *where* $R_{n\times n}$ *denotes the complete ring of* $n\times n$ *matrices over* R ([8], p. 28).

(2.10) P. Hall's Theorem: *If* R *is noetherian and* G *is poly-(cyclic or finite), then* RG *is noetherian* (cf. [1], Theorem 1).

(2.11) L. Small's Theorem: *Let* R *be noetherian,* N *its nilpotent radical. Then the following statements are equivalent*:

(1) $Q(R)$ *exists and is artinian.*

(2) *For each* $x\in R$, x *is regular in* R *if and only if* $x+N$ *is regular in* R/N

(3) *For each* $c\in R$ *with* $c+N$ *regular in* R/N, *there exist* $r\in R$, $n\in N$ *such that* $cr+n$ *is regular in* R. ([9], p. 647).

3. Group-ring with factor sets.

In this section, we will give a general study of group-rings with factor sets. For the sake of convenience, they will simply be called *twisted group-rings.*

DEFINITION 3.1. Let R be a ring and G a group. A pair of mappings (λ,μ) is a *factor set for* G *over* R if λ maps $G\times G$ into the set of non-zero elements of R and μ maps G into the set of all ring-automorphisms of R such that for all $g, h, k \in G, r\in R$, we have

(a) $(r^g)^h\lambda_{h,g}=\lambda_{h,g}r^{hg}$

(b) $\lambda_{g,h}\lambda_{gh,k}=(\lambda_{h,k})^g\lambda_{g,hk}$

(c) $\lambda_{1,h}=\lambda_{g,1}=1$

where $\lambda_{g,h}=\lambda(g,h)$ and $r^g=\mu(g)(r)$.

Now let (λ,μ) a factor set for a group G over a ring R. Let S be the set of all finite formal sums $\sum r_i\bar{g}_i$ with $r_i\in R, g_i\in G$. We define addition in S in the obvious way. We next define multiplication in S as follows: for $a,b\in R$, $g,h\in G$, let

$$(*) \qquad (a\bar{g})(b\bar{h})=ab^g\lambda_{g,h}\overline{gh}.$$

For elements of S which have more than one term, the multiplication is defined by the rule (*) and the distributive laws. Then one verifies that S with addition and multiplication defined as above is a ring having the same identity as R. We will denote S by $R(G;\lambda,\mu)$ to emphasize the role that (λ,μ) plays. We will identify $r\in R$ with $r\bar{1}$ in S and denote $1\bar{g}$ simply by $\bar{g}$.

The following easy facts are useful:

(1) For $r\in R$, $s\in S$, define rs to be the product of r and s as elements of the ring S. Then S becomes a left R-module.

(2) For $g,h\in G$, $\bar{g}\bar{h}=\lambda_{g,h}\,\overline{gh}$ in S.

(3) If both λ and μ are trivial, that is, $\mu(g)(r)=r$ and $\lambda_{g,h}=1$ for all $g, h\in G$, $r\in R$, then S coincides with the ordinary group-ring RG.

We now prove an important theorem which we call the fundamental theorem of twisted group-rings.

THEOREM 3.2. *Let R be a ring and N a normal subgroup of a group G. Then*

$$RG\cong RN(G/N;\ \lambda,\mu)$$

for some factor set (λ,μ) for G/N over RN. Moreover, $\lambda_{g,h}=\lambda(g,h)$ are units in RN for all $g,h\in G$.

PROOF. For each $x\in G/N$, fix an $g_x\in G$ such that $x=Ng_x$. We pick $g_1=1$. Then

$$G=\bigcup\{Ng_x\,|\,x\in G/N\}$$

For $x,y\in G/N$, let $x=Ng_x$, $y=Ng_y$. Then

$$xy=(Ng_x)(Ng_y)=N(g_xg_y).$$

On the other hand,

$$xy=Ng_{xy}.$$

Thus, $g_xg_y=m_{x,y}g_{xy}$ for some $m_{x,y}\in N$. $m_{x,y}$ is uniquely determined, since the representatives $\{g_x\,|\,x\in G/N\}$ are fixed. Define

$$\lambda: G/N\times G/N\rightarrow RN$$

as follows: for $x,y\in G/N$, let

$$\lambda(x,y)=m_{x,y}\in N\subset RN.$$

Next, define

$$\mu: G/N\rightarrow \mathrm{Aut}(RN)$$

as follows: for $x\in G/N$, $\sum r_in_i\in RN$, let

$$\mu(x)(\sum r_in_i)=\sum r_i(g_xn_ig_x^{-1}).$$

Then one verifies that $\mu(x)$ is well defined, $\mu(x)$ is a ring automorphism of RN and (λ,μ) is a factor set for G/N over RN. Let

$$S=RN(G/N;\ \lambda,\mu).$$

Define $\theta: RG\rightarrow S$ as follows: let $r\in R, g\in G$ be given. Then there exist a unique $x\in G/N$ and a unique $n\in N$ such that $g=ng_x$. Thus we define

$$\theta(rg)=(rn)\bar{x}\in RN(G/N;\ \lambda,\mu).$$

We extend θ additively for general elements of RG. Then one verifies that θ is ring isomorphism of RG onto S. This proves the theorem.

LEMMA 3.3. *Let R be an Ore ring and Q its total left quotient ring. Let (λ, μ) be a factor set for G over R. Then there exists a factor set (λ, μ_1) for G over Q such that $\mu_1(g)(r)=\mu(g)(r)$ for all $g \in G$, $r \in R$. (λ, μ_1) is uniquely determined by (λ, μ).*

REMARK. (λ, μ_1) is called an extension of (λ, μ) from R to Q.

Proof of the Lemma. We first note that if such an extension exists, then it is unique, since each $q \in Q$ is of the form $r^{-1}s$ for some $r, s \in R$. Next, let $g \in G$, $q=r^{-1}s \in Q$. Define

$$\mu_1(g) : Q \to Q$$

by putting $\mu_1(g)(q)=[\mu(g)(r)]^{-1}\mu(g)(s)$. We first show that $\mu_1(g)$ is well-defined. Write $\varphi=\mu(g)$, $\varphi_1=\mu_1(g)$. Let $q=r^{-1}s=u^{-1}v$ with $u, v \in R$ and u regular in R. Then $s=ru^{-1}v=p^{-1}kv$, where $ru^{-1}=p^{-1}k$ with $p, k \in R$ and p regular in R. Thus we have $ps=kv$ and $pr=ku$. Then one verifies that $\varphi(r)^{-1}\varphi(s)=\varphi(u)^{-1}\varphi(v)$. Next, for $q_1=r^{-1}s$, $q_2=u^{-1}v$ in Q, write $q_1q_2=(r^{-1}s)(u^{-1}v)=p^{-1}qv$, where $p^{-1}q=r^{-1}su^{-1}$. Thus $r^{-1}s=p^{-1}qu$. Then one easily verifies that $\varphi_1(q_1q_2)=\varphi_1(q_1)\varphi_1(q_2)$.

Let q_1, q_2 be as above. Let $m, n \in R$ with m regular in R be such that $mr=nu$. Then by construction of Q (cf. [4], p. 7), $q_1+q_2=(nu)^{-1}(ms+nv)$. $(nu)^{-1}$makes sense in Q, since $nu=mr$ is regular in R. It is then easy to show that φ_1 is additive.

Finally, one can verify that (λ, μ_1) is a factor set for G over Q, using the following equalities:

$$(r^g)^h\lambda_{h,g}=\lambda_{h,g}r^{hg}$$

and

$$[(r^g)^h]^{-1}\lambda_{h,g}=\lambda_{h,g}(r^{hg})^{-1}.$$

This proves the lemma.

4. Twisted group-rings of finite groups.

In this section, (λ, μ) will always denote a factor set for a group G over a ring R.

LEMMA 4.1. *If R is left artinian and G is finite, then $S=R(G;\lambda, \mu)$ is also left artinian.*

PROOF. Let n be the order of G. We note that as a left R-module, S is isomorphic to the direct sum of n copies of R. Hence S is a left artinian R-module. But R is a subring of S, therefore S is in particular, a left S-module. In other words, S is a left artinian ring.

PROPOSITION 4.2. *Let R be a ring and $Q(R)$ artinian. If G is finite, then $T=R(G;\lambda,u)$ is an Ore ring and $Q(T)$ is artinian.*

PROOF. Let (λ,μ_1) be an extension of (λ,μ) from R to $Q(R)$. Since $Q(R)$ is artinian, so is $S=Q(R)(G;\lambda,\mu_1)$ by Lemma 4.1. It follows that $Q(S)=S$, by (2.3). Next we show that S is a partial left quotient ring of T. This is straightfoward by the common denominator theorem. Hence $Q(T)=Q(S)=S$ is artinian by (2.6). This proves the lemma.

5. Group-rings of poly-(infinite cyclic) groups

LEMMA 5.1. *Let R be a ring without zero divisors and G a poly-(infinite cyclic) group. Then RG has no zero divisors.*

PROOF. In view of Theorem 3.2 and induction, it suffices to show that if G is infinite cyclic and (λ,μ) is a factor set for G over R, then $R(G;\lambda,\mu)$ has no zero divisors. However, the proof of this assertion is similar to the polynomial case and therefore omitted.

LEMMA 5.2. *Let D be a division ring, G the infinite cyclic group and (λ,μ) a factor set for G over D. Then $D(G;\lambda,\mu)$ is a left principle ideal domain and therefore an Ore domain.*

PROOF. Let $S=D(G;\lambda,\mu)$ and x a free generator for G. Let $D[\bar{x}]$ denote the ring of polynomials over D in the indeterminant $\bar{x}$. Then we see that $D[\bar{x}]\subset S$ and for each $y\in S$, $y=x^{-j}f(\bar{x})$ for some integer j and some $f(\bar{x})\in D[\bar{x}]$.

Let A be a non-zero left ideal of S. We can pick a non-zero element $a(\bar{x})\in A\cap D(\bar{x})$ with minimal degree. We claim that $A=Sa(\bar{x})$, the left ideal of S generated by $a(\bar{x})$. Clearly, $Sa(\bar{x})\subset A$. Let $y\in A$, $y\neq 0$. Write $y=x^{-j}f(\bar{x})$ for some integer j and some $f(\bar{x})\in D[\bar{x}]$. Then one sees that $f(\bar{x})\in A$. Next, by the left division algorithm, there exist $q(\bar{x})$, $r(\bar{x})\in D[\bar{x}]$ such that

$$f(\bar{x})=q(\bar{x})a(\bar{x})+r(\bar{x})$$

with $r(\bar{x})=0$ or $\deg(r(\bar{x}))<\deg(a(\bar{x}))$. Thus $r(\bar{x})\in A\cap D[\bar{x}]$, it follows that $r(\bar{x})=0$ by the minimality of $\deg(a(\bar{x}))$. This shows that $f(\bar{x})\in Sa(\bar{x})$. Consequently $y=x^{-j}f(\bar{x})\in Sa(\bar{x})$. Hence $A\subset Sa(\bar{x})$. This proves that $A=Sa(\bar{x})$.

The second assertion follows from the fact that S is a left noetherian domain, so it is semi-prime left Goldie, thus S is an Ore domain. This completes the proof of the Lemma.

PROPOSITION 5.3. *Let R be an Ore domain and G a poly-(infinite cyclic) group. Then RG is again an Ore domain.*

PROOF. We first prove that if G is infinite cyclic and (λ, μ) is a factor set for G over R, then $S=R(G;\lambda,\mu)$ is an Ore domain. We note that S has no zero divisors by Lemma 5.1. Let $D=Q(R)$ be the total left quotient ring of R. Then D is a division ring. Let (λ, μ_1) be an extension of (λ, μ) from R to D. Then $D(G;\lambda,\mu_1)$ is an Ore domain by Lemma 5.2. Let a, b be non-zero elements of $R(G;\lambda,\mu)$. Then there exist $x', y' \in D(G;\lambda,\mu_1)$ such that $x'a=y'b\neq 0$. Write $x'=c^{-1}x$, $y'=d^{-1}y$ for some $c, d \in R$, $x, y \in R(G;\lambda,\mu)$. This is possible by the common denominator theorem. Then $c^{-1}x\,a=d^{-1}yb$ or $dc^{-1}xa=yb$. Write $dc^{-1}=m^{-1}n$, with $m, n \in R$, $m\neq 0$, then we have, $m^{-1}nxa=yb$, or $(nx)a=(my)b\neq 0$, where nx, $my \in R(G;\lambda,\mu)$. This proves that $R(G;\lambda,\mu)$ is an Ore domain.

To finish the proof, use Theorem 3.2 and induction.

PROPOSITION 5.4. *Let R be semi-prime left Goldie and G poly-(infinite cyclic), then RG is again semi-prime left Goldie.*

PROOF. Let $S=Q(R)$ be the total left quotient ring of R. Then S is semi-simple artinian by Goldie's theorem. By the common denominator theorem, one easily verifies that SG is a partial left quotient ring of RG. We want to show that $Q(SG)$ is semi-simple artinian. Let

$$S \cong D_{n\times n}\oplus\cdots\oplus\Delta_{m\times m},$$

where $D, \cdots, \Delta$ are division rings and $D_{n\times n}, \cdots, \Delta_{m\times m}$ are complete rings of $n\times n$, $\cdots, m\times m$ matrices over $D, \cdots, \Delta$ respectively. Thus,

$$\begin{aligned} SG &\cong D_{n\times n}G\oplus\cdots\oplus\Delta_{m\times m}G \\ &\cong (DG)_{n\times n}\oplus\cdots\oplus(\Delta G)_{m\times m}. \end{aligned}$$

All the isomorphisms can be easily established. Now the proposition follows by (2.4), (2.6), (2.8), (2.9) and (5.3).

COROLLARY 5.5. *If $Q(R)$ is semi-simple artinian and G is poly-(cyclic or finite), then $Q(RG)$ exists and is artinian.*

PROOF. The corollary follows by Theorem 3.2, Lemma 4.1 and Proposition 5.3.

We remark that in the above situation, $Q(RG)$ need not be semi-simple. To see this, let R to be a field of characteristic dividing a finite group G. Then $Q(RG)=RG$ and RG is not semi-simple.

6. Proof of the theorem

So far we have applied a different approach to the proof of a slightly more special case of the theorem. However, to finish the proof, we will make use of a technique due to P.F. Smith [9].

LEMMA 6.1. *Let R be artinian and N its nilpotent radical. Let G be poly-(infinite cyclic), then NG is the nilpotent radical of RG.*

PROOF. NG is clearly a nilpotent ideal of RG, so it is contained in the nilpotent radical of RG. On the other hand, since

$$RG/NG \cong (R/N)G$$

and $(R/N)G$ is semi-prime, we have the other inclusion.

LEMMA 6.2. *Let A be an Ore ring and G a group. Let M denote the set of all regular elements of A. Then M is an Ore semi-group of AG.*

PROOF. Clearly every $m\in M$ is a regular element of AG. Let $m\in M$, $x=\sum a_i g_i \in AG$ be given. Then for each i, there exist $m_i\in M$, $b_i\in A$ such that $m_i a_i = b_i m$. Thus, $a_i=m_i^{-1}b_i m$, where $m_i^{-1}\in Q(A)$. For each i, write $m_i^{-1}=m_0^{-1}n_i$ with $m_0\in M$, $n_i\in A$, $m_0^{-1}\in Q(A)$. Then

$$\begin{aligned}\sum a_i g_i &= m_0^{-1}(\sum n_i b_i m g_i)\\ &= m_0^{-1}(\sum n_i b_i g_i)m.\end{aligned}$$

Hence $m_0 x = x'm$, where $x'=\sum n_i b_i g_i \in AG$. This proves the lemma.

Let K be a normal subgroup of a group H such that H/K is infinite cyclic. Let H/K be generated by Kg for some $g\in H$. Let A be a ring (with identity) and $y\in AH$, then $y=\sum a_i g^i$ for suitable integers i and $a_i\in AK$.

LEMMA 6.3. (P.F. Smith [9]). *Let A be semi-prime noetherian, K, H as above and x a regular element of AH. Then there exists $r\in AH$ such that*

$$xr=a_0+a_1g+\cdots+a_l g^l$$

where $a_i\in AK$ for all i and a_l is regular in AK.

LEMMA 6.4. *Let A be artinian and H poly-(infinite cyclic), then $Q(AH)$ exists and is artinian.*

PROOF. Let N be the nilpotent radical of A and $H=H_1 \supset H_2 \supset \cdots \supset H_n \supset H_{n+1}=\{1\}$ be a subnormal series of H such that each factor group H_i/H_{i+1} is infinite cyclic. By induction on n, we may assume that $Q(AH_2)$ exists and is artinian. Let $K=H_2$. We want to show that $Q(AH)$ exists and is artinian. We first note that AH is noetherian by (2.10). Also, $Q(A)=A$ by (2.3). Let $H/K=\langle Kg\rangle$ as before.

By (2.11), it suffices to show that for each $a \in AH$ such that $a+NH$ is regular in AH/NH, there exist $r \in AH$, $n \in NH$ such that $ar+n$ is regular in AH.

Let $f: AH/NH \to (A/N)H$ be the natural isomorphism defined by

$$f((\sum a_i h_i)+NH)=\sum (a_i+N)h_i,$$

where $a_i \in A$, $h_i \in H$ for all i.

Let $$a=\sum a_i h_i, \quad a_i \in A, \quad h_i \in H.$$

Define $$a'=f(a+NH)=\sum (a_i+N)h_i \in (A/N)H.$$

Then a' is regular in $(A/N)H$. Since A/N is semi-prime noetherian, by Lemma 6.3, there exists $r' \in (A/N)H$ such that

$$a'r'=c_0'+c_1'g+\cdots+c_k'g^k,$$

where $c_j' \in (A/N)K$ and c_k' is regular in $(A/N)K$.

Write

$$c_j'=\sum_l \left(a_l^{(j)}+N\right)k_l^{(j)}, \quad j=1,\cdots,k$$

$$r'=\sum_l (\delta_l+N)g_l, \quad \delta_l \in A, \quad g_l \in H.$$

Define

$$c_j=\sum_l \left(a_l^{(j)}k_l^{(j)}\right) \in AK \subset AH,$$

$$r=\sum_l \delta_l g_l \in AH.$$

Then one verifies that

$$\begin{aligned} a'r' &= f(ar+NH) \\ &= f(c_0+c_1g+\cdots+c_kg^k+NH). \end{aligned}$$

Thus

$$ar+n=c_0+c_1g+\cdots+c_kg^k$$

for some $n \in NH$. Since c_k' is regular in $(A/N)K$ and $f(c_k+NH)=c_k'$, one infers that c_k+NH is regular in AH/NH. Thus c_k+NK is regular in AK/NK. (Here we imbed AK/NK into AH/NH by the monomorphism that sends $z+NK$ to $z+NH$ for $z \in AK$. However, as $Q(AK)$ exists and is artinian by assumption, it follows by L. Small's theorem (2.11) that c_k is regular in AK. It is then obvious that $c_0+c_1g+\cdots+c_kg^k$ is regular in AH, as can be easily verified like the polynomial case. This completes the proof of the lemma.

We remark that the technique of the above proof is due to P.F. Smith [9].

Finally we can now prove the following

THEOREM. *If R is an order in an artinian ring and G is a poly-(cyclic or finite), then RG is an order in an artinian ring.*

PROOF. Let H be a normal subgroup of G of finite index in G such that H is poly-(infinite cyclic). Let $A=Q(R)$. Then A is artinian. Denote by M the set of all regular elements of R. Then M is an Ore semi-group in RH by Lemma 6.2. Thus $(RH)_M$ exists. We now show that $(RH)_M=AH$. We note that $Q(AH)$ exists and is artinian by Lemma 6.4. Since $RH \subset AH$, $m^{-1} \in A$ for all $m \in M$, we infer that $(RH)_M \subset AH$. On the other hand, let $y=\sum a_ih_i \in AH$. Write $a_i=m^{-1}b_i$ for all i, with $m \in M$, $b_i \in R$. Then $y=m^{-1}(\sum b_ih_i)$, it follows that $y \in (RH)_M$. This shows that $AH=(RH)_M$.

Since $Q(AH)$ exists and is artinian and $AH=(RH)_M$, one concludes that $Q(RH)=Q(AH)$ and so is artinian.

To finish the proof, use Proposition 4.2 and the fact

$$RG \cong RH(G/H;\ \lambda, \mu)$$

for some factor set (λ, μ) for G/H over RH.

We remark that with section 5 omitted, we have an alternative proof of the theorem. However, section 5 is included here due to the reason that the proof of the special case given there indicates how the classical structure theorems come to play a role and thus it might possess some interest in its own right.

University of Malaya,
Kuala Lumpur,
Malaysia.

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