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Note On The Double Centralizer Of a Module

by

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1. Introduction.

Let R be a ring and M a left R-module. Let $K=\mathrm{Hom}_R(M,M)$. We call K the *centralizer* of R-module M. Clearly M is a right K-module and hence M is a (R,K)-bimodule.

The centralizer of the right K-module M is called the *double centralizer* of M and is denoted by $R^\circ(M)$ (or just R° when there is no risk of confusion). We can easily prove that there exists an embedding of R into the $R^\circ(M)$ when M is faithful over R. In general, R is not isomorphic onto $R^\circ(M)$ even if M is faithful module.

Bourbaki [2] proved that R is dense in $R^\circ(M)$ with respect to finite topology on $R^\circ(M)$, if R is a ring with identity and M is a unital, completely reducible and faithful left R-module.

The purpose of this note is to establish the following results:

If R is a ring with identity and M is a unital, completely reducible and faithful R-module, then

1. Every R-submodule of M is also K-module iff every two distinct irreducible direct summands in a decomposition of M as the direct sum of irreducible submodules are not isomorphic.
2. If M is artinian over K, R is isomorphic onto the double centralizer $R^\circ(M)$.

2. Preliminaries.

PROPOSITION 1. *If M is a faithful R-module, then R is isomorphic onto a subring of $R^\circ(M)$.*

Proof. For an arbitrary $a\in R$, the mapping $A: M\to M$ defined by $Am=am$ is a $K(=\mathrm{Hom}_R(M,M))$-module homomorphism, i.e. $A\in R^\circ(M)$.

Let $\varphi: R\to R^\circ(M)$ be a mapping defined by $\varphi(a)=A$. Then φ is a ring-homororphism. If $\varphi(a)=\varphi(b)$ $am=bm$ for all $m\in M$, that is $a-b\in(0:M)$.

But $(0:M)=0$, since M is faithful R-module. Hence $a=b$. Therefore φ is injective, hence φ is an embedding.

PROPOSITION 2. *Let R-submodule U of M be a direct summand of M. Then*

1. *U is an R°-submodule of M and*
2. *If T is an R-submodule of M, every R-homomorphism of U into T is also an R°-homomorphism of U into T.*

Proof. 1. Let $M=U\oplus U'$ and let $\pi\in K=\mathrm{Hom}_R(M,M)$ be the homomorphism defined by $\pi: m=u+u'\mapsto u$ for U, $u'\in U'$, *i.e.* π is the projection of the R-module M onto the R-submodule U.

Let $a\in R^\circ=\mathrm{Hom}_K(M,M)$. Then $aU=a(M\pi)=(aM)\pi\subseteq U$.

2. Let $\varphi\in\mathrm{Hom}_R(M,M)$ and form $\pi\varphi$. Clearly

$$\pi\varphi\in\mathrm{Hom}_R(U,T)\subseteq\mathrm{Hom}_R(M,M).$$

Hence for an arbitrary $a\in R^\circ$ and each $u=m\pi$ in U,

$a(u\phi) = a(m\pi\varphi) = (am)\pi\varphi = (am\pi)\varphi = (au)\varphi.$

If the R-module M is a direct sum of submodules each isomorphic to a module U, then using Proposition 2, we may show that $R^\circ(M)$ depends only on U and not on the number of summands occurring in the decomposition of M. The following are familiar propositions. We omit the proof of these propositions. [1]

PROPOSITION 3. *Let M be an R-module. The following are equivalent:*

1. $M = \sum_{i\in I} M_i$, *M_i minimal for all $i \in I$.*
   (*Minimal module is a module which has no submodule other than* 0 *and itself.*)
2. $M = \sum_{i\in \lambda} \oplus M_\lambda$, *$M_\lambda$ minimal for all $\lambda \in \Lambda$.*
3. *Every R-submodule U of M is a direct summand.*

PROPOSITION 4. *Let $M = \sum_{i\in I} M_i$, M_i minimal R-modules and let U be a submodule of M. Then there exists $\Lambda \subseteq I$ such that $M = U \oplus \sum_{\lambda\in\Lambda} \oplus M_\lambda$.*

We say an R-module is *completely reducible* if any one of the equivalent condition of Proposition 3 holds, and we say an R-module M is *irreducible* if $RM \neq 0$ and M is minimal.

Let R be a ring with identity. Then every minimal unitary R-module is irreducible. We say an R-module U is *homogeneous of type* I if U is the direct sum of a family $\{M_\lambda \mid \lambda \in \Lambda\}$ of irreducible R-modules M_λ each isomorphic onto the irreducible module I.

Let $M = \sum_{\lambda\in\Lambda} \oplus M_\lambda$ be a decomposition of M as the sum of irreducible submodules. Partition the set $\{M_\lambda \mid \lambda \in \Lambda\}$ into classes $L\rho$, $\rho \in P$ of mutually isomorphic submodules. Let $H\rho = \sum_{M_\lambda \in L_\rho} M_\lambda$. Then clearly

$$M = \sum_{\rho\in P} H\rho.$$

The $H\rho$ are called the *homogeneous components* of M.

PROPOSITION 5. *Let R be a ring with identity. Let U be an irreducible R-submodule of the R-module*

$M = \sum \oplus M_i$, *M_i irreducible.*

Then $U \subseteq M_{i_1} + \cdots\cdots\cdots + M_{i_r}$

where $U \approx M_{i_k}$, $k = 1, 2, \cdots\cdots\cdots, r$.

Conversely, let $U \subseteq M_{i_1} \oplus \cdots\cdots\cdots \oplus M_{i_r}$

and let some element $u \in U$ be expressible as $u = u_{i_1} + \cdots\cdots\cdots + u_{i_r}$, $u_{i_k} \in M_{i_k}$, $k = 1, 2, \cdots\cdots\cdots, r$ *with $u_{j_s} \neq 0$. Then $U \approx M_{i_s}$.*

Proof. Let $0 \neq u \in U$. Then by the definition of sum

$$u = u_{i_1} + \cdots\cdots\cdots + u_{i_r}, \quad 0 \neq u_{i_k} \in M_{i_k}.$$

with uniquely determined $u_{i_1}, \cdots\cdots\cdots, u_{i_r}$ · Now

$$U = Ru \subseteq Ru_{i_1} + \cdots\cdots\cdots + Ru_{i_r}$$

By the irreducibility of the M_i, $Ru_{i_k} = M_{i_k}$ and therefore

$$U \subseteq M_i + \cdots\cdots\cdots + M_{i_k}.$$

The mapping of U to M defined by

$$au \longmapsto au_{i_k}$$

is an isomorphism by the definition of directness of $M = \sum \oplus M_i$ and irreducibility of U. Hence $U \approx M_{i_k}$. The converse is proved similarly.

Let M be a completely reducible R-module and let $\alpha \in K = \mathrm{Hom}_R(M, M)$. Each homogeneous component H of M is mapped into itself by α since an irreducible R-submodule U contained in H is either mapped to 0 or to a submodule isomorphic onto U. By Proposition 5, $H_\alpha \subseteq H$.

In general, however, a submodule $T \neq 0$ of H_1 is not mapped into itself by all $\alpha \in K$. For T contains an irreducible submodule M, (Proposition 4) which may be mapped isomorphically onto any irreducible submoule M_2 in H_1 by some isomorphism φ. This R-isomorphism φ may be extended to a R-homomorphism φ' of M into itself in the following way:

Let $M = \sum_{k \in K} H_k$ and $H_k = \sum_{\rho \in P_k} M_\rho$ M_ρ irreducible.

Let $W = M_2$. Then the mapping φ' defined by

$$H_k \varphi' = 0 \qquad k \neq 1$$
$$M_\rho \varphi' = 0 \qquad \rho \neq 1, \ \rho \in P_1$$
$$m\varphi' = m\varphi \qquad \text{for } m \in M_1$$

is an endomorphism of M. This discussion proves the following characterization of the homogeneous components of a completely reducible R-module.

PROPOSITION 6. *Let the unital R-module M be completely reducible and $K = Hom_R(M, M)$ is its centralizer. Then the homogeneous components of M are the smallest R-submodules H of M which are also right K-modules.*

The following three lemmas are aimed at proving that if M is a completely reducible R-module, then M is also a completely reducible K-module where $K = Hom_R\ (M, M)$.

LEMMA 1. *If R is a ring with identity and M a unital, completely reducible R-module with $M = \sum_{\nu \in J} \oplus M_\nu = \sum_{\nu \in I} \oplus M_\mu$, N_ν, M_μ irreducible, then card$\boldsymbol{I}$=card$\boldsymbol{J}$.*

Proof. If both sets $\boldsymbol{I}$ and $\boldsymbol{J}$ are finite, the lemma is an easy consequence of the Jornan- Hölder theorem concerning chains of submodules of M.

We therefore consider only the case where one of the sets, say $\boldsymbol{J}$, is infinite.

Let $0 \neq X_\mu \in M_\mu$, $\mu \in \boldsymbol{I}$. Then $X_\mu \in \sum_{\nu \in \Lambda} N_\nu$ for some finite subset Λ of $\boldsymbol{J}$. Since M_μ is irreducible, $M_\mu \subseteq \sum_{\nu \in \Lambda} N_\nu$. By directness of the sum, there exists a unique minimal finite subset $\Lambda(\mu)$ of $\boldsymbol{I}$ such that $M_\mu \subseteq \sum_{\nu \in \Lambda(\mu)} N_\nu$.

This determines a mapping $\mu \longmapsto \Lambda(\mu)$. We claim that $\boldsymbol{J} = \bigcup_{\mu \in \boldsymbol{I}} \Lambda(\mu)$. Suppose $\nu_0 \notin \Lambda(\mu)$ for any $\mu \in \boldsymbol{I}$. We have $N_{\nu_0} \subseteq \sum_{i=1}^{k} M_{\mu_i}$ for suitable $\mu_i \in \boldsymbol{I}$. Hence

$$N_{\nu_0} \subseteq \sum \{N_\nu \mid \nu \in \bigcup_{i=1}^{k} \Lambda(\mu_i)\},$$

contrary to the directness of the sum $\sum_{\nu \in J} \oplus N_\nu$. Therefore card$\boldsymbol{I}$ is greater than or equal to the cardinality of a set of finite subsets of whose union is $\boldsymbol{I}$. The later has the same cardinality as $\boldsymbol{J}$. Therefore card$\boldsymbol{I}$ $\geq$card $\boldsymbol{J}$. Reversing the argument we obtain card$\boldsymbol{J}\geq$card$\boldsymbol{I}$ and the lemma is proved.

LEMMA 2. *Let M be a completely reducible R-module, R a ring with identity.*

Then any injective homomorphism of an irreducible submodule M_1 into M may be extended to an automorphism of M.

Proof. Let $M = \sum_{\rho \in P} \oplus H_\rho$ be the decomposition of M into its homogeneous components and suppose $M_1 \subseteq H_1$. Let σ_1 be an isomorphism of M_1 onto M_1' say. M_1' is also contained in H_1. Now by Proposition 4

$$H_1 = (\sum_{\lambda \in \Lambda} \oplus M_\lambda) \oplus M_1$$

and

$$H_1=(\sum_{\lambda'\in\Lambda}\oplus M_{\lambda'})\oplus M_{1'}.$$

By Lemma 1 cardΛ=cardΛ'. There is therefore a one-one corespondence between Λ and Λ', say $\lambda\longmapsto\lambda'$. Let σ_λ: $M_\lambda\to M_{\lambda'}$ be an isomorphism of M_λ onto $M_{\lambda'}$ and let σ denote the resulting isomorphism of H_1 onto H_1. Clearly σ extends σ_1. Now define τ by

$$m_1\tau=m_1\sigma,\quad m_1\in H_1$$

$$m_i\tau=m_i,\quad m_i\in H_i,\quad i\neq 1.$$

τ extends σ (and hence σ_1) and is an automorphism of M.

LEMMA 3. *Let* $1\in R$ *and let M be a unital, completely reducible left R-module, K its centralizer. If* $m\neq 0$ *belongs to an irreducible submodule U of M, mK is an irreducible right K-module.*

Proof. Let $0\neq m'\in mK$. Then there is $\alpha\in K$ such that $m'=m\alpha$. The mapping $u\longmapsto u\alpha$ of U onto U_α is an isomorphism. By Lemma 2 this may by extended to an automorphism β of M. Therefore $m'=m\beta$ and $m'\beta^{-1}=m$ showing that $m'K=mK$.

PROPOSITION 7. *Let* $1\in R$ *and let M be a unital, completely reducible R-module. Then M is also a completely reducible K-module,* $K=Hom_R\ (M, M)$.

Proof. $M=\sum mK$ where m ranges over all nonzero m belonging to irreducible R-submodules.

3. Main theorems.

LEMMA 4. *Let R be a ring and M be a completely reducible R-module.*

Then an arbitrary submodule of M is also comptelely reducible

Proof. Let $M=\sum_{\lambda\in\Lambda}\oplus M_\lambda$, M_λ irreducible. Then by Proposition 4, there exists $P\subseteq\Lambda$ such that $M=U\oplus\sum_{\rho\in P}\oplus\ M\rho$. Let $I=\Lambda-P$. Then $U=\sum_{i\in I}\oplus M$.

THEOREM 1. *Let R be a ring with identity and M be a unital and completely reducible left R-module.*

Then M is represented as the direct sum of irreducible submodules $\{M_\lambda|\lambda\in\Lambda\}$ *each of two submodules of which are not isomorphic, iff every left R-module of M is right K-submodule where* $K=Hom_R\ (M, M)$.

Proof. 1. Suppose $M=\sum_{\lambda\in\Lambda}\oplus M_\lambda$, M_λ irreducible, each two submodules in $\{M_\lambda|\lambda\in\Lambda\}$ are not isomorphic over R. Then M_λ is right K-submodule of M by Proposition 6. Let N be a left R-submodule of M. Then

$$N=\sum_{\rho\in P}\oplus M_\rho \text{ for some } P\subseteq\Lambda,$$

i.e. $N=\sum_{\rho\in P}M\rho$. Hence N is a right K-submodule.

2. Suppose $M=\sum_{\lambda\in\Lambda}\oplus M_\lambda$, M_λ irreducible and $M_i\approx M_j$ for some $i\neq j$ in Λ.

Let $\varphi:M_i\to M_j$ be the isomorphism of M_i onto M_j.

The mapping $\varphi':M\longrightarrow M$ defined by

$$M_\lambda\varphi'=0 \quad \text{for } \lambda\neq i$$

$$m\varphi'=m\varphi \quad \text{for } m\in M_i$$

is an endomorphism of M and hence $\varphi'\in K$. Since $M_i\varphi'=M_i\varphi=M_j\neq M_i$, the left R-submodule M_i is not right K-submodule.

LEMMA 5. *If M is an artinian and completely reducible R-module, then M is a finite direct sum of irreducible R-submodules.*

Proof. Suppose $M=\sum_{\lambda\in\Lambda}\oplus M_\lambda$ where M_λ are irreducible and Λ is infinite set. Then

$$M=\sum_{i=1}^{\infty}\oplus M_i\supset\sum_{i=2}^{\infty}\oplus M_i\supset\sum_{i=3}^{\infty}\oplus M_i\supset\cdots\cdots\cdots$$

is an infinite strictly descending chain of R-submodules of M where is a subset of $\{M_\lambda \mid \lambda\in\Lambda\}$. Hence M is not artinian.

THEOREM 2. *Let R be a ring with identity and let M be a unital, faithful and completely reducible left R-module. If M is artinian over K where* $K=Hom_R\ (M, M)$, *R is isomorphic onto the double centralizer* $R^\circ(M)=Hom_K(M, M)$,

Proof. For an arbitrary $a\in R$ the mapping $\alpha: M\longrightarrow M$ defined by $\alpha m=am$ is a K-homomorphism. Hence $\alpha\in R^\circ(M)$. Let $\varphi: R\longrightarrow R^\circ(M)$ be a mapping defined by $\varphi(a)=\alpha$. Then φ is a injective ring-homorphism of R to $R^\circ(M)$ by Proposition 1, since M is faithful R-module.

We will prove that φ is surjective.

By Proposition 7, M is a completely reducible K-module. Since M is an artinian right K-module, M is a finite direct sum of irreducible right K-submodules by Lemma 5. Let

$$M=M_1\oplus M_2\oplus\cdots\cdots\cdots\oplus M_n,\quad M_i \text{ irreducible over K.}$$

Let $0\neq x_i\in M_i$, then $x_iK=M_i$ since K has the identity and M_i are irreducible right K-submodules. Hence

$$M=x_1K\oplus\cdots\cdots\cdots\oplus x_nK.$$

Let $T=M\oplus\cdots\cdots\cdots\oplus M$, the direct sum of n copies of M. To a given $\alpha\in R^\circ(M)$ we define a mapping A of T into itself by

$$A(m_1, m_2, \cdots\cdots\cdots, m_n)=(\alpha m_1,\ \alpha m_2, \cdots\cdots\cdots \alpha m_n).$$

Then $A\in R^\circ(T)$.

Now let X denote the R-submodule of T generated by the element $(x_1, x_2, \cdots, x_n)$. T is clearly completely reducible R-module and therefore, by Proposition 3, X is a direct summand of T. By Proposition 2, X is also an $R^\circ(T)$-submodule of T. Therefore $A(x_1,\ x_2, \cdots\cdots\cdots,\ x_n)\in X$. But $X=\{(rx_1,\ rx_2, \cdots\cdots\cdots,\ rx_n) \mid r\in R\}$. Therefore there exists $a\in R$ such that

$$(ax_1,\ ax_2, \cdots\cdots\cdots,\ ax_n)=A(x_1, x_2, \cdots\cdots\cdots,\ x_n)=(\alpha x_1,\ \alpha x_2, \cdots\cdots\cdots,\ \alpha x_n).$$

Now let x be an arbitrary element of K-module M. Then there exist $k_i\in K$ such that

$$x=x_1k_1+x_2k_2+\cdots\cdots\cdots+x_nk_n, \text{ and then}$$

$$\begin{aligned}\alpha x&=\alpha(x_1k_1+x_2k_2+\cdots\cdots\cdots+x_nk_n)\\&=(\alpha x_1)k_1+(\alpha x_2)k_2+\cdots\cdots\cdots+(\alpha x_n)k_n\\&=(ax_1)k_1+(ax_2)k_2+\cdots\cdots\cdots+(ax_n)k_n\\&=a(x_1k_1+x_2k_2+\cdots\cdots\cdots+x_nk_n)\\&=ax\end{aligned}$$

Hence $\varphi(a)=\alpha$. Therefore φ is a ring-isomorphism of R onto $R^\circ(M)$.

References

(1) Ernst-August Behrens, Ring Theory, Academic Press, 1972.

(2) Bourbaki, N., "Algebra", Chapter 8, Modules et anneaux semi-simples, Hermann, Paris, 1958.

(3) Jacob Barshay, Topics in Ring Theory, W.A. Benjamin, inc. 1969.