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On Tightness of Product Space.

by

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A.H. Stone is introduced that the topological product of a metric space and a normal countable compact space is normal. [1]

A space is called strongly countable compact if the closure of every countable subset has compact closure. [2]

If a space is strongly countable compact, then it is countable compact.

If it is countable compact, then it is pseudocampact.

Neither of these impliation is reversible in general.

The propertys of being strongly countable compact has very interesting properties. It is productive, closed hereditary, and preserved under continuous transformation. The next theorem gives an additional result for this properties. The following Lemma is well known. [3]

It can be shown that every linearly ordered countable compact space is strongly countable compact and $\sum$-products of bicompact provide another example of strongly countable compact spaces.

V. I. Malyhin, in his paper several theorems on the relationship between the cardinal invariant of the space expX and the properties of the space X are proved, i. e. introduced the concepts of tightness of the space X. [2].

In this note. I tried to prove that product of normal strongly countable compact space and a paracompact of countable tightness is collectionwise normal. Some notation and terminologies are due to reference [2], [3].

[Lemma 1] If X is normal and $\{F_1, F_2, \cdots\cdots\cdots, F_k\}$ is finite collection of closed subsets of X, then $\bigcap_{i=1}^{k} cl_{\beta X}F_i = cl_{\beta X}[\bigcap_{i=1}^{k} F_i]$

[Lemma 2] If X is normal and $F: 2^X \longrightarrow 2^{\beta X}$ is defined by $F(K) = cl_{\beta X}\ K$, then F is an imbedding.

[Theorem 3] If X is normal and strongly countable compact, then 2^X is strongly countable compact.

[Definition 4] The tightness of a space is less than or equal to m if every set containing the closure of each subset of cardinality at most m is closed. i. e. $t(X)$, the tightness of the space X, is defined as follows: $t(X) \leq m$, if the fact that $x \in [A]$ implies that there exists $B \subset A$, such that $|B| \leq m$ and $x \in [B]$. [2].

[Lemma 5] Let the tightness of the space X be countable and let Y be strongly countably compact.

Then the projection $p: X \times Y \rightarrow X$ is a closed mapping.

[Proof] Let U be a open in $X \times Y$ and let $S = \{x \in X : p^{-1}(x) \subseteq U\}$.

We will show that S is open. Otherwise, we could find a point $x \in S$ in the closure of some countable set $\{x_n \in X \sim S : n = 1, 2, \cdots\cdots\cdots\}$

For each n we choose $y_n \in Y$ so that the point (x_n, y_n) is not in U and fix a basic system of nbds of the point $x : \sigma = \{V_\xi : \xi \in \Sigma\}$.

For every $\xi \in \Sigma$ we define the countable set $R_\xi = \{y_n : x_n \in V_\xi\}$.

The bicompacts R_ξ, $\xi \in \Sigma$ form a centend system.

Let $y \in \bigcap \{R^\xi : \xi \in \Sigma\}$. The nbds $V_{\xi 0} \in \sigma$ and H of the points x and y, respectively, are such that $V_{\xi_0} \times H \subseteq U$. We can find $y_{n0} \in R_{\xi_0} \cap H$.

One sees easily that $(x_{n_0}, y_{n_0}) \in U$. This contradiction proves the Lemma. Therefore we obtain the following theorem.

[Theorem 6] The product of a paracompact of countable tightness and a normal strongly countably compact space is collectionwise normal.

[Remark]

1) The term bicompact was used to denote a compact set and the term compact was used to denote a sequentially compact.
2) $x \in \beta N - N$ (N is countable discrete space). $X = N \cup \{x\}$ and $Y = {}_\beta N - \{x\}$. Then the tightness of X is obviously countable, the space Y is countable comnact and the projection of the product $X \times Y$ onto X is not a closed mappings.
3) The product of a bicompact with a normal strongly countable compact Frécht-Uryson space (which is sequential) does not have to be normal [5]. We also note that the product of a first-countable paracompact with a seperable metric space does not have to be normal [4].

요 약

거리공간과 Normal countable compact의 位相積이 Normal이라는 것은 A.H. Stone에 의하여 이미 밝혀졌고, V.I. Malyhin은 space expX의 Cardrnal invariant와 공간 X 사이의 관계를 논하였다. 본 논문에서는 V.I. Malyin이 밝힌 tightness의 개념을 도입하여 countable tightness의 pracompact와 normal strongly countable compact 공간의 topological product가 Normal이라는 것을 증명하였다.

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