EXTERME POINTS OF THE SHELL OF A LINEAR RELATION

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Let \( A \) be a bounded linear operator defined on a complex Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \). Let \( W(A) = \{ \langle Ax, x \rangle : \|x\| = 1, \ x \in H \} \) be the numerical range of \( A \). For each complex number \( z \), let \( M_z \) denote the subset of \( H \), \( \{ x \in H : \langle Ax, x \rangle = z \|x\|^2 \} \). In [5], M. Embry characterized the extreme points of \( W(A) \) in terms of \( M_z \). We obtain an analogy of her result in the setting of the shell \( s(A) \) of a linear relation \( A \) in \( H \) (Theorem). In preparing for the proof of our main result, we also get several lemmas which might be useful in their own right.

The notion of the shell \( s(A) \) of a linear relation \( A \) in a Hilbert space \( H \) was introduced by C. Davis in [1], as a solid in the three dimensional Euclidean space \( \mathbb{R}^3 \). To get familiar with the tools and terminologies which will be used later, we review first some rudiments of [1], [2].

Let \( \mathbb{C} \) denote the extended complex plane, \( \mathbb{C} \cup \{ \infty \} \), and \( B \) the unit ball of \( \mathbb{R}^3 \). Let \( \zeta, h \) be a complex number and a real, respectively.

We define a mapping \( \theta : B \to \mathbb{C} \), by sending \( (\zeta, h) \) to the point \( z \in \mathbb{C} \) such that \( (\zeta, h) \) is located on the line passing through the point \((z, 0)\) and the north pole \((0, 1)\) of \( B \). That is, \( \theta(\zeta, h) = \frac{\zeta}{\bar{h}} \), \( h \neq 1 \) and \( \theta(0, 1) = \infty \).

Let \( S \) denote the unit sphere \( \{ (\zeta, h) \in \mathbb{R}^3 : |\zeta|^2 + h^2 = 1 \} \). The stereographic projection \( \tau : \hat{C} \to S \) is defined as follows: \( \tau(z) = \left( \frac{2z}{1 + |z|^2}, -\frac{1 + |z|^2}{1 + |z|^2} \right), \ z \in \mathbb{C} \) and \( \tau(\infty) = (0, 1) \).

Note that \( \theta(\tau(z)) = z \), for all \( z \in \hat{C} \).

A M"obius transformation \( \mu : \hat{C} \to \hat{C} \) is defined by sending \( z \in \mathbb{C} \) to \( \mu(z) = \frac{az + b}{cz + d}, \) where \( ad - bc \neq 0 \) and \( \mu(\infty) = \frac{-a}{c} \). This leads us to define the Möbius transformation, again denoted by \( \mu : S \to S \) by sending \( \tau(z) \) to \( \tau(\mu(z)) \), for \( z \in \mathbb{C} \). If we put \( \tau(z) = (\zeta, h) \), \( \mu(\tau(z)) = (\zeta', h') \), then the coordinates are related by the following matrix equation (p. 77 [1]).

\[
\begin{bmatrix}
1 + h' \\
\zeta' \\
1 - h'
\end{bmatrix} =
\begin{bmatrix}
a \bar{a} & \bar{b} & b \bar{a} & \bar{b} \\
a \bar{c} & \bar{d} & b \bar{c} & d \bar{b} \\
\bar{c} & \bar{d} & \bar{d} & \bar{d}
\end{bmatrix}
\begin{bmatrix}
1 + h \\
\zeta \\
1 - h'
\end{bmatrix}.
\]

Now if we apply the above equation (1) to the points \( (\zeta, h), (\zeta', h') \) of the unit ball \( B \), where \( 1 \)'s are replaced by \( \sqrt{|\zeta|^2 + h^2} \), we still get a mapping, also called the Möbius transformation \( \mu \) of \( B \) onto itself, which sends \( (\zeta, h) \) to \( (\zeta', h') \). In the case \( d = \bar{a}, \ c = -\bar{b} \), the Möbius transformation \( \mu \) is just a typical rigid rotation of the unit ball \( B \).
Let $A$ be a linear relation in $H$, that is, a linear subspace of $H \oplus H$. The shell $s(A)$ of $A$ is defined as the set all points

$$\left\{ \left( \frac{2<y,x>}{\|y\|^2+\|x\|^2}, \frac{-\|x\|^2+\|y\|^2}{\|x\|^2+\|y\|^2} \right) : (y,x) \in A, (y,x) \neq (0,0) \right\}.$$  

(p. 70, Definition 1.1 [1]). If $\dim(H) \geq 3$, then $s(A)$ is a convex subset of the unit ball $B$ (p. 304 Theorem 10.1 [2]). Let $I = \{(y,x) \in A : y = x\}$. The point spectrum $\sigma_p(A)$ of $A$ is defined by $\sigma_p(A) = \{z \in C : (A-zI) \cap \{(0,0) \oplus H\} \neq \{(0,0)\}\}$, with $\infty$ adjoined if $0 \in \sigma_p(A^{-1})$, where $A^{-1} = \{(x,y) \in H \oplus H : (y,x) \in A\}$. The approximate point spectrum $\sigma_a(A)$ of $A$ is the set $\{z \in C : \exists \,(y_m,x_m) \in A, \exists \, n \geq 1 \text{ and } \|y_m\| \to 0 \}$. (p. 71 Definitions 2.1–2.5, Proposition 2.1 [1]). Then we have

$$\bigcap \sigma(A) = \tau(\sigma_p(A))$$  

and

$$\bigcap \mathcal{S}(A) = \tau(\sigma_a(A))$$

where $\mathcal{S}(A)$ denote the closure of $s(A)$ in $\mathbb{R}^2$. The numerical range $W(A)$ of $A$ is defined as the set $\{<y,x> : \|x\| = 1, (y,x) \in A\}$, with $\infty$ adjoined in the case $\infty \in \sigma_a(A)$ (p. 73 Definition 3.1 [1]). It is easy to see that

$$\theta(s(A)) = W(A)$$  

and

$$\mu(W(A)) = W(\mu(A)).$$

For the $\mu$ as above, the Möbius transformation $\mu(A)$ of a subset $A$ of $H \oplus H$ (i.e., a relation $A$ in $H$), is defined by

$$\mu(A) = \{(ay+bx, cy+dx) : (y,x) \in A\}$$  

(p. 77 [1]).

For a linear relation $A$, we have

$$\mu(s(A)) = s(\mu(A))$$  

(p. 78 Theorem 5.1 [1]).

The next lemma was obtained by Embry (pp. 647–648, Lemma 1[5]). We state it here without proof.

**Lemma 1.** Let $A$ be a bounded linear operator on a Hilbert space $H$. For each complex number $\lambda$, denote $M_{\lambda} = \{x \in H : <Ax,x> = \|x\|^2\}$. Let $z$ be in the interior of a line segment with end points $a$ and $b$ in $W(A)$. Let $\epsilon$ be the largest number $\epsilon$ (in the open interval $0,1$) and a complex number $\alpha$, $|\alpha| = 1$ such that $tx + (1-t)\alpha \in M_\lambda$ and $sx - (1-s)\alpha \in M_\lambda$. Consequently, $M_{\lambda} \subseteq M_z + M_z = A$.

The above lemma is extended easily to the case of a linear relation $A$.

**Lemma 2.** Let $A$ be a linear relation in a Hilbert space $H$, with $\infty \notin \sigma(A)$. For each complex number $\lambda$, denote $Y_\lambda = \{(y,x) \in A : <y,x> = \lambda \|x\|^2\}$. Let $z$ be in the interior of a line segment with end points $a$ and $b$ in the numerical range $W(A)$ of $A$. Then $Y_z \subseteq Y_a + Y_b = A$.
Proof. Let \((y_1, x_1) \in Y_a\) and \((y_2, x_2) \in Y_b\) with \(\|x_2\| = 1\). We want to show that \((y_2, x_1) \in Y_a + Y_b\). Since \(\infty \in \sigma_p(A)\), we may assume that \(x_1 \neq 0\). Also, since \(Y_a, Y_b + Y_a\) are homogeneous, we can still assume that \(\|x_1\| = 1\). A simple computation shows that \(x_1\) and \(x_2\) must be linearly independent, by using the fact that \(\infty \in \sigma_p(A)\). We consider the Hilbert space \(H_1\) spanned by \(x_1, x_2, y_1\) and \(y_2\). Then we find a linear operator \(A_1\) on \(H_1\) such that \(A_1 (x_i) = y_i, i=1, 2\). By applying the previous Lemma 1, we see easily that \((y_2, x_1) \in Y_a + Y_b\) and that \(Y_b + Y_a\) is a linear subspace of \(A\), ie. \(A = Y_a + Y_b\) (cf. p. 648 Proofs of Lemma 1, Theorem 1 (iii), [5]). Q.E.D.

Corollary 3. Let \(A\) be a linear relation in \(H\) with \(\infty \in \sigma_p(A)\). Let \(z \in W(A)\) and \(Y_z\) be as in the above lemma. If \(Y_z\) is a linear subspace of \(A\), then \(z\) is an extreme point of \(W(A)\) (cf. p. 647 Theorem I(i) [5]).

Proof. The proof is similar with that of Theorem 1(i), p. 648 [5] and omitted. Q.E.D.

Lemma 4. Let \(A\) be a linear relation in a Hilbert space \(H\). Let \(\mu\) be a Möbius transformation of \(A\) onto another relation \(A'\) in \(H\), by \((y, x) \rightarrow (ay + bx, cy + dx), ad - bc \neq 0\). Then the following hold.

(i) \(A'\) is also a linear relation and \(\mu\) is a topological linear isomorphism on \(A\) onto \(A'\), with respect to the norms \(\|(y, x)\| = \|y\| + \|x\|\), \((y, x) \in A\) and also for \((y, x) \in A'\).

(ii) If \(A\) is closed, so is \(A'\).

Proof. The verifications are elementary and omitted. Q.E.D.

The next lemma is also considere as a natural generalization of theorem 1 (ii), p. 647 and Lemma 2, p. 648 [5]. But our proof appears more translucent in the new setting of the linear relation.

Lemma 5. Let \(A\) be a linear relation in a Hilbert space \(H\) with \(\infty \in \sigma_p(A)\). As in Lemma 2, let \(Y_L = \{(y, x) \in A : \langle y, x \rangle = \lambda \|x\|\} for a complex number \(\lambda\). Let \(z \in W(A)\) and \(L\) be a supporting line of \(W(A)\) through \(z\). Then the following hold.

(i) \(A_L = \bigcup \{ Y_L : \lambda \in L \} \subset W(A)\) is a linear subspace of \(A\).

(ii) If \(z\) is an extreme point of \(W(A)\) then \(Y_z\) is a linear subspace of \(A\).

(iii) \(A_L = A\) if and only if \(W(A) \subseteq L\).

(iv) If \(A\) is closed, so are \(A_L\) and \(Y_z\).

Proof. (i) Note that \(\infty \in W(A)\), since \(\infty \in \sigma_p(A)\). We can find a suitable affine transformation \(\mu\) of the plane such that the following is true. \(\mu(W(A))\) is contained in the closed left half-plane, with respect to the imaginary axis, \(\mu(L)\) is the imaginary axis and \(\mu(z) = 0\), the origin of the plane. Note that \(\mu(W(A)) = W(\mu(A))\), by the identity (5). Let \([\mu(A)]\) denote the set of all \((y, x) \in H \oplus H\) such that \((y, u), (v, x) \in \mu(A)\) for some \(u, v \in H\). Clearly \(\mu(A) \subseteq [\mu(A)]\). We consider the real valued functional \(f\) on \([\mu(A)]\), defined by \(f(y, x) = \text{Re} \langle y, x \rangle\), where \((y, x) \in [\mu(A)]\). Note that \(f\) is a bilinear form on \([\mu(A)]\) with respect to the real scalar multiplication. Let \(\mu(A)_1 = \{(y, x) \in \mu(A) : \text{Re} \langle y, x \rangle = 0\} = \{(y, x) \in \mu(A) : f(y, x) = 0\}.\) We claim that \(\mu(A)_1\) is a linear subspace of \(\mu(A)\). Let \((y_i, x_i) \in \mu(A)_1, i = 1, 2\). Then \(f(y_1 + y_2, x_1 + x_2) = f(y_2, x_1) + f(y_1, x_2) = 0\).
Similarly $f(y_1-y_2, x_1-x_2) = -f(y_2, x_1) - f(y_1, x_2) \leq 0$.

If follows that $f(y_1 + y_2, x_1 + x_2) = 0$, proving that $\mu(A)$ is linear. But a simple computation shows that $\mu(A_1) = \mu(A)$. Therefore $A_1$ is linear as well, by Lemma 4 (i).

(ii) Let $Z$ be an extreme point of $W(A)$. We consider $\mu(A) = \{(y, x) \in \mu(A) : <y, x> = 0\}$, where $\mu$ is as in the proof of (i) above.

Since $\mu(Y_2) = \mu(A)$, it only needs to show that $\mu(A)_0$ is a linear subspace of $\mu(A)$. But $\mu(A)_0 = \{(y, x) \in \mu(A) : \text{Im} <y, x> = 0\}$. Since 0 is an extreme point of $\mu(A)$, we see that $\text{Im} <y, x> \leq 0$, for all $(y, x) \in \mu(A)_1$ or $\text{Im} <y, x> \geq 0$, for all $(y, x) \in \mu(A)_1$. Let $[\mu(A)]$ be similarly defined as $[\mu(A)]$ above. We consider again a real bilinear form $g$ on $[\mu(A)]$ by defining $g(y, x) = \text{Im} <y, x>$, for $(y, x) \in [\mu(A)]$. By the same procedure as for $\mu(A)$ and $f$ above, we can conclude that $\mu(A)_0$ is linear.

(iii) Obvious. (iv) It follows from Lemma 4 (ii). Q.E.D.

The necessity implication of the next proposition was overlooked in [5] even for a bounded operator $A$.

**Proposition 6.** Let $A$ be a linear relation in a Hilbert space $H$ with $\sigma_P(A)$. Let $Y_i$ denote as in the above Lemma 5. Define $A_1 = \bigcup \{ Y_i : \lambda \in L \cap W(A) \}$. Then $A_1$ is linear if and only if $L$ is a supporting line of $W(A)$ through $z$.

**Proof.** We only need to prove the necessity. First observe that every point $\lambda \in L \cap W(A)$ can not be located in the interior of a line segment whose end points $a, b$ are in $W(A)$ and $a \in L \cap W(A)$. For, if it were, then $Y_a \subset Y_k + Y_l \subset A_1 + A_1 = A_1$, by Lemma 2, a contradiction. Q.E.D.

**Lemma 7.** Let $A$ be a linear relation in a Hilbert space $H$. Let

$$\zeta(y, x) = \frac{2<y, x>}{||x||^2 + ||y||^2}, \quad h(y, x) = \frac{-||x||^2 + ||y||^2}{||x||^2 + ||y||^2} \quad \text{and} \quad s(y, x) = (\zeta(y, x), h(y, x)) \in B, \text{ the unit ball of } \mathbb{R}^2, \text{ for } (y, x) \in A, (y, x) \neq (0, 0).$$

Let $\beta$ be the uniquely determined number, $0 \leq \beta \leq \infty$ such that $\sup \{ h(y, x) : (y, x) \in A \sim \{(0, 0)\} \} = -\frac{1 + \beta^2}{1 + \beta^2}$, that is, $\beta$ is the norm $||A||$ of $A$ (of P. 81 Definition 7.1 [1]). Then the following hold.

(i) Let $h_1 = -\frac{1 + \beta^2}{1 + \beta^2}$. Then the set $A_1 = \{(y, x) \in A : (-||x||^2 + ||y||^2) = h_1(||x||^2 + ||y||^2)\}$ is a linear relation.

(ii) If $A$ is closed, so is $A_1$.

**Proof.** (i) If $h_1 = 1$, namely $\beta = \infty$, then the proof is obvious. Let $-1 \leq h < 1$. It is immediate to see that $A_1 = \{(y, x) \in A : ||y|| = \beta||x||\}$, and $||y|| \leq \beta||x||$ for all $(y, x) \in A$. Now let $(y_i, x_i) \in A$, $i = 1, 2$. By the parallelogram law, $||y_1 + y_2||^2 + ||y_1 - y_2||^2 = 2||y_1||^2 + 2||y_2||^2 = 2\beta^2(||x_1||^2 + ||x_2||^2) = \beta^2(||x_1 + x_2||^2 + ||x_1 - x_2||^2)$. Therefore, $||y_1 + y_2||^2 = \beta^2||x_1 + x_2||^2 + \beta^2||x_1 - x_2||^2$, since $||y_1 - y_2|| \leq \beta||x_1 - x_2||$. But $||y_1 + y_2|| \leq \beta||x_1 + x_2||$. It follows that $||y_1 + y_2|| = \beta||x_1 + x_2||$ and $(y_1, x_1) + (y_2, x_2) \in A_1$.

(ii) Straight-forward. Q.E.D.

Our main theorem is an analogy of Theorem 1 (i) [5] of Embry.

**Theorem.** Let $A$ be a linear relation in a complex Hilbert space $H$ of dimension $\geq 3$. 

Sa-Ge Lee
For each \( u = (\zeta, h) \in s(A) \), where \( \zeta \) is a complex number, \( h \) a real, let \( Y_{u} = \{ (y, x) \in A : 2\langle y, x \rangle = \zeta (\|x\|^2 + \|y\|^2) \text{ and } -\|x\|^2 + \|y\|^2 = h (\|x\|^2 + \|y\|^2) \} \). Suppose that \( u \) is a boundary point of \( s(A) \) then \( u \) is an extreme point of \( s(A) \) if and only if \( Y_{u} \) is a linear subspace of \( H \).

Proof. In the case that \( u \in S \), the unit ball, the assertion can be proved easily by the identity (2). Now let \( u \in S \). Let \( L \) denote a supporting plane of \( s(A) \) through \( u \). We draw a straight line from the origin of \( S \) to the direction of the open halfspace determined by \( L \), that does not meet with \( s(A) \), such that the line is also perpendicular to \( L \). Let \( v \) be the intersection of \( S \) with this line. Then \( v \in s(A) \). Let \( \mu \) be a Möbius transformation of \( B \) which brings \( v \) to the north pole of \( B \), by a rigid rotation. Then \( \mu(v) \in s(\mu(A)) \) by (7).

Now let \( L' \) denote the plane rotated from \( L \) by \( \mu \). Clearly \( L' \) is a supporting plane of \( s(\mu(A)) \) at \( \mu(u) \) and it is parallel to the complex plane. Let \( Y_{w} = \{ (0, 0) \} \cup \{ (y, x) \in \mu(A) : s(y, x) = w \in s(\mu(A)) \} \). Let \( \mu(A)_{1} = \{ Y_{w} : w \in L' \cap s(\mu(A)) \} \).

By Lemma 7 (i), \( \mu(A)_{1} \) is a linear subspace of \( \mu(A) \). Note that \( s(\mu(A)_{1}) = L' \cap s(\mu(A)) \). Let \( Y = \bigcup \{ Y_{w} : w \in L \cap s(A) \} \).

We claim that \( \mu(Y_{u}) = \mu(A)_{1} \) and

\[ \mu(Y_{u}) = Y_{\mu(u)} \quad (8) \]

We shall only verify (8). Let \( (y, x) \in Y_{u} \), so \( s(y, x) = u \) (See Lemma 7 for the notation of \( s(y, x) \). Then \( \mu(u) = \mu(s(y, x)) = s(\mu(u)) \), by (6). It follows that \( \mu(y, x) = Y_{\mu(u)} \).

Now \( W(\mu(A)_{1}) = \theta(s(\mu(A)_{1})) \), by (4). Note that \( \theta \) is a one to one correspondence which sends a line segment to a line segment. Let \( Y_{\theta(\mu(u))} = \{ (y, x) \in \mu(A)_{1} : \langle y, x \rangle = \theta(\mu(u)) \} \).

It is immediate to see that

\[ Y_{\mu(u)} = Y_{\theta(\mu(u))} \quad (9) \]

Now we have the following chain of equivalent statements from (a) to (g).

(a) \( u \) is an extreme point of \( s(A) \).
(b) \( \mu(u) \) is an extreme point of \( s(\mu(A)) \).
(c) \( \mu(u) \) is an extreme point of \( s(\mu(A)_{1}) \).
(d) \( \theta(\mu(u)) \) is an extreme point of \( W(\mu(A)_{1}) \).
(e) \( Y_{\mu(u)} \) is linear (Corollary 3, Lemma 5 (ii) and the above (9)).
(f) \( Y_{u} \) is linear (the identity (8)). Q.E.D.

Remark. Let \( u \) be an arbitrary point of \( s(A) \) in the above theorem. Question: If \( Y_{u} \) is linear, must \( u \) lie on the boundary of \( s(A) \)? We conjecture that the answer is positive.

References


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