COMPACT HYPERSURFACES WITH ANTINORMAL
(f, g, u, v, \lambda)-STRUCTURE IN
AN ODD-DIMENSIONAL SPHERE

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Dedicated to professor Chung-Ki Pahk on his sixeth birthday

0. Introduction.

It is well known that a hypersurface of an almost contact metric manifold or of an odd-dimen-
sional sphere with canonical contact structure admits an (f, g, u, v, \lambda)-structure, that is, there exist a set of a tensor field \( f \) of type (1, 1), a Riemannian metric tensor \( g_{ij} \), two 1-forms \( u_i \) and \( v_i \) (or two vector fields \( u_i = u_i^k g_{kj} \), \( v_i = v_i^k g_{kj} \)) and a function \( \lambda \) which satisfy

\[
\begin{align*}
\delta f = -f + u_i u^i + v_i v^i, \\
\delta f = -f + u_i u^i + v_i v^i, \\
f f_i u_i = \lambda v_i, \\
fv_i = -\lambda u_i, \\
\lambda = v_i v^i = 1 - \lambda^2, \\
u v_i = 0
\end{align*}
\]

([8], [9]).

In studying the manifold structure of hypersurfaces of the sphere admitting the induced
(f, g, u, v, \lambda)-structure many authors have found several results (cf. [1], [3], [4], [5], [7]
and [9]).

Recently Blair, Ludden and Yano in their paper [1] proved that

**THEOREM A.** Let \( M \) be a complete orientable hypersurface in \( S^{2n+1}(1) \) such that induced
(f, g, u, v, \lambda)-structures satisfy

\[
f f_i h^i + h f_i = 0,
\]

\( h^i \) being second fundamental tensor of \( M \).

If the scalar curvature \( K \) of \( M \) is constant and \( \lambda(1 - \lambda^2) \) is non-zero almost everywhere function on \( M \), then \( M \) is isometric to \( S^n \) or \( S^n \times S^n \).

The main purpose of the present paper is reformulation of Theorem A. We shall prove this theorem under the condition that the hypersurface is compact and the sectional curvature of the section spanned by two vectors \( u \) and \( v \) has semi-definite sign, but without constancy of scalar curvature.

In section 1 and 2, we prepare fundamental properties of the hypersurface with (0.2)
in an odd-dimensional sphere. In the last section 3, we study the compact hypersurfaces.
under the condition stated above by using the same method as in the paper [5] and results of Simons and of Chern–do Carmo–Kobayashi.

1. Preliminaries.

Let $S^{2n+1}(1)$ be a $(2n+1)$-dimensional sphere of radius 1 in Euclidean $(2n+2)$-space $E^{2n+2}$. As is well known, $S^{2n+1}(1)$ admits a canonical contact metric structure $(\phi, \xi, \eta, g)$, which is induced from the natural Kaehlerian structure equipped on $E^{2n+2}$. Throughout this paper, manifolds, functions, vector fields and other geometric objects we discuss are assumed to be differentiable and of class $C^\infty$.

Let $M$ be a $2n$-dimensional orientable and connected hypersurface in $S^{2n+1}(1)$ covered by a system of coordinate neighborhood $\{U: x^i\}$, where here and in the sequel, indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$ and the summation convention will be used with respect to these indices.

It is well known that the $(f, g, u, v, \lambda)$-structure induced on a hypersurface $M$ of $S^{2n+1}(1)$ with induced Riemannian metric $\tilde{g}$ and the second fundamental tensor $h_{ij}$ satisfy

\begin{align}
\nabla_j f^h_i &= -g_{ji} \omega^h + \delta_j^h u_i - h_{ji} \omega^h + h_{ij} \omega^h, \\
\nabla_{ij} &= f_{ij} + \lambda h_{ij}, \\
\nabla_j v_i &= -h_{ji} f^i_j + \lambda g_{ji}, \\
\n\nabla_j \lambda &= h_{ji} \omega^i - v_j
\end{align}

(cf. [7], [9]), where $f^h_i = h_{ij} g^{h'}$, $(g^h) = (g_{ij})^{-1}$, $f^i_j$, $u_i$ and $v_i$ are components of $f$, $u$ and $v$ respectively, $\nabla_j$ being the operator of covariant differentiation with respect to $g_{ij}$.

Moreover, the structure equations of the submanifold $M$, i.e. the equations of Gauss and Codazzi are given by

\begin{align}
\nabla_j h_{ij} &= -\delta_j^k g_{ki} - \delta_j^k g_{ki} + h_{ki} h_{ij} - h_{ij} h_{ki}, \\
K_{ij} &= \nabla_j h_{ij} - \nabla_j h_{ji} = 0
\end{align}

respectively.

From (1.5), the scalar curvature $K$ of $M$ is written in the form

\begin{align}
K &= 2n(2n-1) + (h_{ij})^2 - h_{ij} \omega^i
\end{align}

in terms of the second fundamental form.

As a matter of convenience, in the sequel, we denote by $N_0 = \{P \in M | \lambda(P) = 0\}$, $N_1 = \{P \in M | \lambda^2(P) = 1\}$ and $N = M - (N_0 \cup N_1)$.

First we note that (1.2) implies the set $N_1$ is bordered (cf. [3], [4]).

We assume on the submanifold $M$ of $S^{2n+1}(1)$ the $(f, g, u, v, \lambda)$-structure is antinormal, that is,

\begin{align}
S_{ij} &= 2v_j (\nabla_j \omega^h - \lambda \omega^h) - 2v_i (\nabla_i \omega^h - \lambda \omega^h),
\end{align}

or, equivalently
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(1.9) \[ h_{ji} f^i - h_k f_j = 0 \]

because \(N_1\) is a bordered set (See [9]).

**Remark.** If the hypersurface \(M\) in \(S^{2n+1}(1)\) satisfies

(1.10) \[ h_{ji} f^i - h_k f_j = 2\phi f_{ij} \]

or, equivalently

(1.11) \[ \phi_j = \phi_{ij} \]

by virtue of (1.3), \(\phi\) being a function on \(M\). We know that \(\phi\) is a constant on \(M\) if \(n > 2\) (See [5]).

If we put

(1.12) \[ T_{ij} = h_{ji} - \phi g_{ij} \]

then (1.6) and (1.10) become respectively

(1.13) \[ \phi_j = \phi_{ij} \]

(1.14) \[ T_{ij} f^i - T_{ij} f^j = 0 \]

Comparing (1.6) and (1.9) with (1.10)-(1.14), the condition (1.9) seems to be more essential than (1.10).

2. Antinormal \((f, g, u, v, \lambda)\)-structures induced on \(S^{2n+1}(1)\).

We first prove

**Lemma 2.1.** If \(M\) is a hypersurface with antinormal \((f, g, u, v, \lambda)\)-structure \((n > 1)\), then the set \(N_0\) is a bordered set.

**Proof.** If there exists a connected open kernel \(W\) of \(N_0\),

(2.1) \[ h_{ji} v^i = v_j \] on \(W\)

by virtue of (1.4).

Transvecting (1.9) with \(f^i v^j\) and using (0.1) and (2.1), we find

(2.2) \[ h_{kt} v^t = u_k + \alpha v_k \] on \(W\),

where we have put \(\alpha = h_{ji} v^i v^j\).

Differentiating (2.2) covariantly and substituting (1.2) and (1.3) with \(\lambda = 0\), we have

\[ (\phi_j h_{ji}) v^i - h_j h_{kt} f^i = f_{ij} + (\phi_j \alpha) v_j - \alpha h_{kt} f^i \] on \(W\),

from which, taking skew-symmetric parts and using (1.6),

\[ -2h_j h_{kt} f^i = 2f_{ij} + (\phi_j \alpha) v_j - (\phi_j \alpha) v_k + \alpha (h_{ji} f^i - h_{kt} f^j) \]

or, using (1.9)

\[ 2h_j h_{kt} f^i = 2f_{ij} + (\phi_j \alpha) v_j - (\phi_j \alpha) v_k \] on \(W\).

Transvecting the last equation with \(v^t\) and taking account of (2.2), we find \(\phi_j \alpha\)
Thus we have
\[ h^j_i h^i_k = g_{ji} + 2\mu_i u_j + (2 + \alpha^2) v_j v_i + \alpha (v_i u_j + u_i v_j) \]
on \( W \).

On the other hand, when one gives attention to equation (2.1) and (2.2), it is seen that there exists, at any point of \( W \), two eigenvectors of the second fundamental tensor of \( M \) belong to the plane section \( P(u, v) \) spanned by \( u^k \) and \( v^k \). Let \( \sigma_1 \) and \( \sigma_2 \) be eigenvalues corresponding to the eigenvectors. Then the eigenvalues are roots of the quadratic equation: \( \sigma^2 - \alpha \sigma - 1 = 0 \), and consequently \( \sigma_1 = \frac{1}{2} (\alpha + \sqrt{\alpha^2 + 4}) \), \( \sigma_2 = \frac{1}{2} (\alpha - \sqrt{\alpha^2 + 4}) \).

For an eigenvector \( X \) corresponding to the eigenvalues \( \sigma_1 \), (1.9) implies \( h^k_i (f^j_i X^i) = \sigma_1 (f^k_i X^i) \). This means that the transforms \( f^k_i X^i \) of the vector \( X \) by the linear transformation \( f \) is also an eigenvector, of which eigenvalue is equal to \( -\sigma_1 \).

From (2.3) we see that there does not exist eigenvalue corresponding eigenvector \( Y \) orthogonal to the plane section \( P(u, v) \).

Thus we have \( \sigma_1 = -\sigma_2 \) and consequently \( \alpha = 0 \). Therefore, (2.3) becomes
\[ h^j_i h^i_k = -g_{ji} + 2(\mu_i u_j + v_i v_j) \quad \text{on} \quad W, \]
from which, \( 0 \leq h^j_i h^i_k = -2(n-2) \) on \( W \), and consequently \( h_{ji} = 0 \) on \( W \). But \( h_{ji} \) can not be zero on \( W \) by virtue of (2.1). So \( W \) is empty. This completes the proof of Lemma 2.1.

**Lemma 2.2.** Under the same assumptions as those stated in Lemma 2.1, we have
\[ h^j_i u^i = \beta v_j, \quad (2.4) \]
\[ h^j_i v^i = \beta u_j, \quad (2.5) \]
\[ h^i_i = 0 \quad (2.6) \]
and
\[ v^i = (\beta - 1) v_j \quad (2.7) \]
in \( N \cup N_0 \), where \( \beta \) is a function in \( N \cup N_0 \) defined by \( \beta = \frac{1}{1-\lambda} h_{ki} u^i v^k \).

**Proof.** Transvecting (1.9) with \( f^i_k \), we find
\[ h_{ri} f^j_i f^i_r + h_{jk} - (h^j_ri u_k) u_i - (h^j_i v^i) v_k = 0, \]
from which, taking skew-symmetric parts,
\[ (h^j_ri u_k) u_i - (h^j_ri u^i) u_k + (h^j_k v^i) v_k - (h^j_i v^i) v_j = 0. \]
Since \( u_i \) and \( v_i \) do not vanish in \( N \cup N_0 \), transvecting the equation above with \( u^i \) and \( v^i \) and using (0.1), we have
\[ h^j_i u^i = \alpha u_j + \beta v_j, \quad (2.8) \]
\[ h^j_i v^i = \beta u_j + \gamma v_j \quad (2.9) \]
Compact hypersurfaces with antinormal \((f, g, u, v, \lambda)\)-structure in an odd-dimensional sphere respectively in \(N \cup N_0\), where \(\alpha\) and \(\gamma\) are defined by
\[
\alpha = \frac{1}{1 - \lambda^2} h_u u', \quad \gamma = \frac{1}{1 - \lambda^2} h_v v'.
\]

Differentiating (2.8) covariantly, we have\[ (\mathcal{V}_u h_j) u_j + h_j^i (\mathcal{V}_u u_i) = (\mathcal{V}_u \alpha) u_j + (\mathcal{V}_u \beta) v_j + \alpha \mathcal{V}_u u_j + \beta v_j \]
in \(N \cup N_0\). Taking skew-symmetric parts and using (1.2), (1.3), (1.6) and (1.9), we obtain
\[
(2.10) \quad (\mathcal{V}_u \alpha) u_j - (\mathcal{V}_u \beta) v_j - (\mathcal{V}_j \beta) v_i = 2 \alpha f_{jk},
\]
from which, we see that \(\mathcal{V}_u \alpha\) and \(\mathcal{V}_j \beta\) are linear combinations of \(u_j\) and \(v_j\), that is,
\[
(1 - \lambda^2) \mathcal{V}_u \alpha = (w \mathcal{V}_u \alpha) u_j + (w \mathcal{V}_u \beta) v_j,
\]
\[
(1 - \lambda^2) \mathcal{V}_j \beta = (v \mathcal{V}_u \alpha) u_j + (v \mathcal{V}_u \beta) v_j.
\]
Thus (2.10) reduced to
\[
(2.11) \quad (v \mathcal{V}_u \alpha - u \mathcal{V}_u \beta) (u_j v_j - u_j v_i) = 2 \alpha f_{jk},
\]
in \(N \cup N_0\).

On the other hand, \(f_{ji}\) is of rank \(2n - 2 \geq 2\) in \(N \cup N_0\) by assumption \(\dim M \geq 2\). Thus we have, from the equation above, \(\alpha = 0\) and consequently \(u \mathcal{V}_u \beta = 0\) in \(N \cup N_0\). From this fact (2.10) becomes
\[
(2.11) \quad \mathcal{V}_j \beta = \frac{1}{1 - \lambda^2} (v \mathcal{V}_u \beta) v_j.
\]

Transvecting (1.9) with \(w v_i\) and using (2.8) and \(\alpha = 0\), we find \(\lambda (1 - \lambda^2) \gamma = 0\), i.e. \(\lambda \gamma = 0\) in \(N \cup N_0\). But, since \(N_0\) is bordered set and \(N \cup N_0\) is open, \(\gamma = 0\) in \(N \cup N_0\). Thus (2.4) and (2.5) are obtained.

Transvecting (1.9) with \(f^i\) and taking account of \(\alpha = \gamma = 0\), we get (2.6). Therefore, Lemma 2.2 is proved.

**Lemma 2.3.** Under the conditions as those stated in Lemma 2.1, the equations
\[
(2.12) \quad (1 - \lambda^2) \mathcal{V} \beta = -2 \lambda \beta (\beta + 1) v_j
\]
and
\[
(2.13) \quad (1 - \lambda^2) (h_j h^i + \beta g_{ji}) = \beta (\beta + 1) (u_j u_i + v_j v_i)
\]
hold in \(N \cup N_0\).

**Proof.** Differentiating (2.5) covariantly and using (1.2) and (1.3), we have in \(N \cup N_0\)
\[
(\mathcal{V}_u h_{ji}) v^t + h^t (-h_{s} f^i + \lambda g_{ik}) = (\mathcal{V}_u \beta) u_j + \beta (f_{jk} - \lambda h_{jk}),
\]
from which, taking skew-symmetric parts and taking account of (1.6), (1.9) and (2.11),
\[
(2.14) \quad 2 h^t h_{ji} f^i = 2 \beta f_{jk} + \frac{1}{1 - \lambda^2} (v \mathcal{V}_u \beta (u_j v_j - u_j v_i)).
\]

Transvecting (2.14) with \(u^t\) and using (2.4) and (2.5), we see that
Thus (2.12) is gotten from (2.11) and (2.15).
Substituting (2.15) into (2.14), we find in $N \cup N_0$
\[ h'_j h_{kl} = \beta f_k u_j + \frac{\lambda}{1-\kappa^2} \beta (\beta+1) (u_j u_k - u_k u_j), \]
from which, transvecting $f^i_1$ and using (2.4) and (2.5), we find (2.13).

3. Compact hypersurfaces with antinormal $(f, g, u, v, \lambda)$-structure.

In this section, we assume that $M$ is a compact hypersurface with antinormal $(f, g, u, v, \lambda)$-structure in $S^{2n+1}$, $(n \geq 1)$.

From (2.4), (2.5) and (2.13) we get at most four distinct principal curvatures $\pm \sqrt{-\beta}$, $\pm \beta$ at each point in $N \cup N_0$, and hence $N_0$ and $N_1$ being bordered sets, also in $M$ because of the continuity of principal curvatures. Under the condition (1.9) we see easily that the multiplicities of $\beta$ and $-\beta$ are 1 and those of $\sqrt{-\beta}$ and $-\sqrt{-\beta}$ are $n-1$.

We have from (2.13)
\[ k_j k^i = 2\beta (\beta - n + 1) \quad \text{in} \quad N \cup N_0. \]
Thus, by (1.7), (2.6) and (3.1), the scalar curvature $K$ is given by
\[ K = 2(\beta + n) (2n - 1 - \beta) \quad \text{in} \quad N \cup N_0. \]
Since $\beta$ is non-positive, solving the quadratic equation above with respect to $\beta$, we have $\beta = (n-1-\sqrt{\theta})/2$ in $N \cup N_0$, where $\theta = (n-1)^2 - 4(K/2 - n(2n-1)) \geq 0$ in $N \cup N_0$. We have $\theta > 0$ in $M$, because $N_1$ is bordered. Hence we can define a function $\bar{\beta}$ in $M$ by
\[ \bar{\beta} = (n-1-\sqrt{\theta})/2. \]
Then, the function $\bar{\beta}$ thus defined is an extension of the function $\beta$ defined only in $N \cup N_0$ and differentiable on $M$. Thus all differential equations involving $\beta$ established in $N \cup N_0$ are valid about $\bar{\beta}$. Without fear of confusion, we denote the extension $\bar{\beta}$ by the same letter $\beta$. Then $\beta(x)$ is equal to 0 or $-1$ at each point $x$ in $N_1$ (See [5]).

For a symmetric matrix $H$ of degree $2n$, it is well known that
\[ 2n \text{tr} H^2 - (\text{tr} H^2 - (\text{tr} H)^2 - (\text{tr} H) (\text{tr} H)^3 = \sum_{i<j} (\sigma_i - \sigma_j)^2 (1+\sigma_i \sigma_j), \]
where $\sigma_i$ are eigenvalues of $H$, (cf. [6]).

We have already known that the principal curvatures at each point of $M$ are $\sigma_1 = \beta$, $\sigma_2 = -\beta$, $\sigma_3 = \sqrt{-\beta}$ and $\sigma_4 = -\sqrt{-\beta}$, and their multiplicities are respectively 1 and $n-1$.

In [5], a formula of Simon's type was computed without consideration of these multiplicities. But if we consider these multiplicities, then
\[ \sum_{i<j} (\sigma_i - \sigma_j)^2 (1+\sigma_i \sigma_j) = (\sigma_1 - \sigma_2)^2 (1+\sigma_1 \sigma_2) + (n-1) (\sigma_1 - \sigma_3)^2 (1+\sigma_1 \sigma_3). \]
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\[
+(n-1)(\sigma_2 - \sigma_3)^2 (1+\sigma_2 \sigma_3) + (n-1)^2 (\sigma_3 - \sigma_4)^2 (1+\sigma_3 \sigma_4)
= 4\beta(\beta+1)(\beta-n+1)(n-\beta).
\]

Of course, from (3.4), calculating the left hand side of (3.4) by using (2.6) and (3.1), we also get the same result.

Now, using this result, if we calculate a formula of Simon's type for the hypersurface of constant mean curvature in a sphere \([6]\), then

\[
(3.6) \quad \frac{1}{2} \Delta(h_{\jmath k}h^{\jmath k}) = (\nabla \lambda)(\nabla^k \lambda) + 4\beta(\beta+1)(\beta-n+1)(n-\beta).
\]

On the other hand, differentiating both sides of (2.12), we find

\[
-2(\nabla \lambda)(\nabla \lambda) + (1-\lambda^2)\nabla \lambda
= -2(\nabla \lambda)\beta(\beta+1)v_j - \lambda(2\beta+1)(\nabla \lambda) v_j
= 2\lambda(\beta+1)(\nabla \lambda) v_j
\]

on \(M\), from which, using (1.3), (2.7) and (2.12),

\[
(3.7) \quad \Delta \beta = \frac{2\lambda(\beta+1)}{1-\lambda^2} \{\lambda^2(5\beta-2n+1)+2\lambda(1-\beta)(1+\beta)\} \quad \text{on } M.
\]

From (3.1) and (3.2), we find

\[
\Delta(h_{\jmath k}h^{\jmath k}) = 4(\nabla \lambda)(\nabla \lambda) + 2(2\beta-n+1)\Delta \beta,
\]

from which, substituting (2.12) and (3.7),

\[
(3.8) \quad \Delta(h_{\jmath k}h^{\jmath k}) = \frac{\Delta \beta(\beta+1)}{1-\lambda^2} \{4\lambda^2 \beta(\beta+1) + (2\beta-n+1)
\times [2\lambda^2(\beta-n+2)+(1-\beta)(1-\lambda^2)]\}.
\]

Comparing (3.6) and (3.8), we obtain

\[
(\nabla \lambda)(\nabla^k \lambda) = \frac{\Delta \beta(\beta+1)}{1-\lambda^2} \{4\lambda^2 \beta(\beta+1) + (2\beta-n+1)
\times [2\lambda^2(\beta-n+2)+(1-\beta)(1-\lambda^2)]\},
\]

from which \(\beta(\beta+1)\geq 0\) at every point in \(N_0\) since \(\beta\) is nonpositive.

On the other hand, from (0.1), (1.5), (2.4) and (2.5), we have

\[
K_{ijkl} h^{ij} h^{kl} = -(1-\beta^2)(1-\lambda^2)\lambda^2,
\]

from which sectional curvature of the section \(P(u,v)\) is \(1-\beta^2\). Now we assume that the sectional curvature is semi-definite sign in \(NU N_0\), that is, \(1-\beta^2\) is semi-definite sign.

At first we consider the case of \(1-\beta^2\geq 0\). Then, since \(\beta\) is non-positive, \(-1\leq \beta\leq 0\) in \(NU N_0\).

As we have already shown that \(\beta(\beta+1)=0\) at every point in \(N_1\), \(\beta(\beta+1)\leq 0\) on the whole space \(M\). Hence from (3.1)

\[
\text{tr } h^2 - h_{\jmath k}h^{\jmath k} \leq 2n.
\]
Next, in the case of $1 - \beta \leq 0$ in $NU \cup N_0$, we also find $\beta(\beta + 1) \geq 0$ at every point in $M$ by the same reason. Then, by continuity of $\beta$, it follows that $\beta$ is not greater than $-1$. But, from the compactness of the space $M$, we can prove that $\beta = -1$ on $M$ by the same method in the last part of the paper [5]. Hence in both cases we have from (3.1) that $\text{tr} \, h^2 \leq 2n$. Since $M$ is compact, by the result of Simons, $\text{tr} \, h^2$ can take only two values 0 and $2n$. So, combining Chern–do Carmo–Kobayashi’s result we have

**THEOREM.** Let $M$ be a compact hypersurface with antinormal $(f, g, u, v, \lambda)$–structure in an odd–dimensional sphere $S^{2n+1}(1)$. If sectional curvature of the section spanned by mutually orthogonal vectors $u$ and $v$ is semi–definite sign, then $M$ is a great sphere $S^{2n}$ or $S^n \times S^n$.

**Bibliography**


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