ON QUASI-SEMIDEVELOPABLE SPACES

By Il Hae Lee

1. Introduction.

In recent years there have been several studies concerning the generalization of developable spaces. H. R. Bennett [3] defines a quasi-developable space which is useful to obtaining metrization theorems for $M_i$-spaces ($i=1, 2, 3$). C. C. Alexander [1] introduced semi-developable spaces and proves that a space is semi-metrizable if and only if it is semi-developable $T_0$-space. He also introduces cushioned pair semidevelopable spaces [2] to obtain a generalization of Morita's metrization theorem.

In the paper [8] we have generalized developable spaces further and introduced a quasi-semidevelopable space which includes both classes of semi- and quasi-developable spaces. We denote a quasi-semidevelopable space by a qs-developable space as in [8]. Thus we have shown that

(1) A space is semi-developable if and only if it is qs-developable and perfect.
(2) In a qs-developable space hereditary $\chi_1$-compactness, hereditary Lindelöf property and hereditary separability are equivalent.

(3) A separable regular $T_0$-space with a point-finite qs-development is metrizable.

In the present paper we extend some of the results appeared in [8]. A qs-stratifiable space is defined and show that it is semi-stratifiable if it is perfect. The closure preserving property for the qs-developable space is investigated to find relations between qs-developable spaces and $M_i$-spaces. By doing so, we get an example which shows that a regular closure preserving semi-developable space is not always metrizable. We give more general definition of cushioned pair qs-development than the one appeared in [8]. Thus we obtain a generalization of theorem 2.3 of [8].

By a space we will mean a topological space in this paper. We assume every topological space is $T_1$ unless otherwise mentioned. We adopt the convention if $G$ is a subset of a topological space $X$, then $\text{Int}(G)$ denotes the interior of $G$ in $X$ and $\text{cl}(G)$ denotes the closure of $G$ in $X$. If $G$ is a collection of sets, then $G^* = \bigcup \{g | g \in G\}$. Finally $N$ denotes the set of all positive integers. All undefined terms are as in [7].

2. Quasi-semidevelopable spaces.

Let $\gamma = (\gamma_1, \gamma_2, \cdots)$ be a sequence of collections of subsets of a topological space $(X, \tau)$. Consider following three conditions of the sequence $\gamma$:

(1) For each $x \in X \{\text{St}(x, \gamma_n) | n \in N, \quad x \in \gamma_n^*\}$ is a local basis at $x$.
(2) Each $\gamma_n$ is a covering of $X$.
(3) Each $\gamma_n$ is a subclass of $\tau$.

The condition (1) is equivalent to the following:

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(a) For each \( x \in X \) and for each positive integer \( n \) \( St(x, r_n) \) is a neighborhood of \( x \) provided \( St(x, r_n) \neq \emptyset \), and

(b) For each \( x \in X \) and for each open set \( U \) containing \( x \) there exists a positive integer \( n \) such that \( x \in St(x, r_n) \subseteq U \).

If \( \gamma \) satisfies all the above three conditions (1), (2) and (3), then \( \gamma \) is called a development for the space \( X \). And \( \gamma \) is a semi-development for \( X \) if it satisfies only the two conditions (1) and (2) \([1]\). On the other hand if \( \gamma \) satisfies the conditions (1) and (3), then it is called a quasi-development for \( X \) \([3]\).

We generalize these spaces and define a new class of spaces.

**Definition 2.1.** A sequence \( \gamma = (\gamma_1, \gamma_2, \cdots) \) of collections of subsets of a space \( X \) is a quasi-semidevelopment for \( X \) if \( \gamma \) satisfies the condition (1).

A space is said to be quasi-semidevelopable if it has a quasi-semidevelopment.

From the definition it is clear that semi-developable spaces and quasi-developable spaces are qs-developable.

G. D. Creede \([6]\) introduced a class of semi-stratifiable spaces as a generalization of semi-metric spaces. He shows that a \( T_1 \)-space is semi-metric if and only if it is first countable and semi-stratifiable. The class of semi-stratifiable spaces contains \( M_3 \)-spaces \([4]\). (For the definition of \( M_3 \)-spaces see section 3.) Now we introduce a class of qs-stratifiable spaces as a generalization of qs-developable spaces and show that a qs-stratifiable space is semi-stratifiable if it is perfect.

**Lemma 2.2.** A space \( X \) is \( T_1 \) and qs-developable if and only if there is a mapping \( g: \mathbb{N} \times X \to P(X) \) such that

1. the set \( Z_x = \{ n \in \mathbb{N} \mid g(n, x) \neq \emptyset \} \) is infinite,
2. the collection \( \{ g(n, x) \mid n \in Z_x \} \) is a local basis at the point \( x \),
3. for each \( i, k \in Z_x \), \( g(i, x) \subseteq g(k, x) \) if \( i < k \),
4. if \( x \in g(n, x) \) for every \( n \in Z_x \), then \( x \) is a cluster point of the sequence \( \langle x_n \rangle \), and
5. if \( x \in g(n, y) \), then \( x \in g(n, x) \).

**Proof.** Let \( \gamma = (\gamma_1, \gamma_2, \cdots) \) be a qs-development for the space \( X \). We may assume that the set \( \{ n \mid x \in \gamma_n \} \) is infinite for each point of \( X \). Let \( f: \mathbb{N} \times X \to P(X) \) be the mapping such that \( f(n, x) = St(x, \gamma_n) \) and \( Z_x = \{ n \in \mathbb{N} \mid f(n, x) \neq \emptyset \} \). Then clearly the set \( Z_x \) is infinite. In order to get a mapping which satisfies the above conditions we define a mapping \( g \) as follows:

\[
g(n, x) = \begin{cases} \bigcap_{i \in Z_x} f(i, x) & \text{if } n \in Z_x' \\ \emptyset & \text{if } n \notin Z_x' \end{cases}
\]

The set \( Z_x = \{ n \in \mathbb{N} \mid g(n, x) \neq \emptyset \} \) is infinite since \( Z_x = Z_x' \). Clearly \( \{ f(n, x) \mid n \in \mathbb{N} \text{ and } x \in \gamma_n \} \) is a local basis at \( x \) for each \( x \in X \), and so is \( \{ g(n, x) \mid n \in Z_x \} \). The third property of the lemma is clear.

To show (4), let \( x \in g(n, x) \) for every \( n \in Z_x \). Then it is easily seen that \( x_n \) is in every \( g(n, x) \) for \( n \in Z_x \). Since the sequence of sets \( \{ g(n, x) \} \) is decreasing, \( x \) is a cluster
point of the sequence \( \langle x_n \rangle \).

To show the last property, let \( x \in g(n, y) \). Then \( x \in \bigcap_{i \in \mathbb{Z}_+} f(i, y) \) which is a subset of \( f(n, y) \) where \( n \in \mathbb{Z}_+ \). Therefore \( x \in ST(y, r_n) \). This implies that \( y \in f(n, x) \) and \( n \in \mathbb{Z}_+ \).

Now we prove the converse. Let \( \gamma_n = \{ \{ x, y \} \mid y \in g(n, x) \wedge x \in g(n, y) \} \).

Each \( \gamma_n \) is a collection of subsets of \( X \) which has two elements. For the sequence \( \langle \gamma_n \rangle \) it can be shown that \( \{ ST(x, \gamma_n) \} \) is a local basis of \( x \). Thus the collection \( \langle \gamma_1, \gamma_2, \ldots \rangle \) is a \( q_s \)-development for \( X \).

The above lemma motivates us to formulate the concept of \( q_s \)-stratifiable spaces by a slight modification of the necessary condition of lemma 1.5. We state the formal definition as follows:

**Definition 2.3.** A topological space \( (X, \tau) \) is said to be \( q_s \)-stratifiable if there exists a mapping \( f : N \times X \to \tau \) such that
1. \( Z_x = \{ n \mid f(n, x) \neq \emptyset \} \) is infinite.
2. \( x \) belongs to the intersection of \( f(n, x) \) for all \( n \) in \( Z_x \).
3. If \( x \) belongs to \( f(n, x_n) \) for every \( n \) in \( Z_x \), then \( x \) belongs to the closure of \( \{ x_1, x_2, \ldots \} \) and
4. \( n \) is an element of \( Z_x \) whenever \( x \in f(n, y) \).

If we set \( f(n, x) = \text{Int}(g(n, x)) \), where \( g(n, x) \) is that of lemma 2.2, then clearly \( f(n, x) \) satisfies all the above conditions. Therefore every \( q_s \)-developable space is \( q_s \)-stratifiable.

**Lemma 2.4.** A space is \( q_s \)-stratifiable if and only if there is an open covering \( \langle O_n \rangle \) of \( X \) such that
1. for each \( x \) there exist infinitely many \( O_n \) which contain \( x \) and
2. for each open set \( U \) there corresponds a sequence of closed sets \( \langle U_n \rangle \) such that \( U = \bigcup_x (U_n \cap O_n) \)

and

\[ U_n \subset V_n \quad \text{if} \quad U \subset V. \]

**Proof.** Suppose \( X \) is \( q_s \)-stratifiable under the mapping \( f : N \times X \to \tau \). Let \( O_n = \bigcup_{x \in X} f(n, x) \).

Then clearly \( \bigcup_n O_n = X \). Furthermore each \( O_n \) is characterized by the set \( Z_x \), namely,

\[ \bigcup_{x \in X} f(n, x) = \{ x \in X \mid x \in f(n, x) \}. \]

For an open set \( U \) let

\[ U_n = X - \bigcup_{x \in X - U} f(n, x). \]

Then \( \langle U_n \rangle \) is a sequence of closed sets. It is not difficult to verify that \( U_n = \bigcup (U_n \cap O_n) \).

The remaining part of the theorem is an easy consequence of the fact that \( \bigcup_{x \in X - U} f(n, x) \)
To prove the converse we set \( f(n, x) = (X - (X - x_n)) \cap O_n \).

Then \( f : N \times X \to X \) and satisfies (1) to (4) of definition 2.3.

**Theorem 2.5.** A space is semi-stratifiable if and only if it is qs-stratifiable and perfect. (For the definition of a semi-stratifiable space see [6].)

**Proof.** Suppose \( X \) is qs-stratifiable and perfect. Let \( U \) be an open set. Then by lemma 2.4 there is a sequence of closed sets \( \{U_n\} \) and an open covering \( \{O_n\} \) of \( X \) such that \( U = \bigcup_n (U_n \cap O_n) \). Since \( X \) is perfect,

\[
U = \bigcup_n (U_n \cap \bigcup F_n) = \bigcup U_m
\]

where each \( F_m \) is closed and \( U_m = U \cap F_m \).

Let \( U \) and \( V \) be open sets. There correspond two sequences of closed sets \( \{U_n\} \), \( \{V_n\} \) respectively such that \( U_n \subseteq V_n \) for each \( n \). It follows that \( U_m \subseteq V_m \) since \( (U_n \cap F_m) \subseteq (V_n \cap F_m) \). Hence \( X \) is semi-stratifiable. The converse is evident by Theorem 1.2. of [6].

If a space \( X \) is first countable, there is a mapping \( h : N \times X \to P(X) \) such that \( \{h(n, x) | n \in \mathbb{N}\} \) is a decreasing local basis. Moreover if the space \( X \) is qs-stratifiable by the mapping \( f \), then \( \{g(n, x) | n \in \mathbb{N}\} \) is clearly a local basis of \( X \) where

\[
g(n, x) = f(n, x) \cap h(n, x).
\]

The mapping \( g \) also satisfies all conditions of definition 2.3. Thus we know that a first countable qs-stratifiable space is qs-developable.

**3. \( M_i \)-spaces \((i=1, 2, 3)\) vs. qs-developable spaces.**

Let \( \gamma \) be a collection of subsets of a space. For every subclass \( \gamma' \) of \( \gamma \) if

\[
\text{cl}(\bigcup \{C \in \gamma' : C \in \gamma\}) = \bigcup \text{cl}(C),
\]

then \( \gamma \) is said to be closure preserving. A space is closure preserving qs-developable if each \( \gamma_n \) is closure preserving where \( \gamma = (\gamma_1, \gamma_2, \ldots) \) is a qs-development for \( X \).

A regular space \( X \) is said to be an \( M_i \)-space if \( X \) has a \( \sigma \)-closure preserving basis. An \( M_2 \)-space is a regular space which has a \( \sigma \)-closure preserving quasi-basis [5].

**Definition 3.1.** If \( \gamma \) and \( \delta \) are collections of subsets of \( X \), we say that \( \gamma \) is cushioned in \( \delta \) if there exists a mapping \( D : \gamma \to \delta \) such that

\[
\text{cl}(\bigcup \{C \in \gamma' : C \in \gamma\}) = \bigcup \text{cl}(D(C))
\]

for every subclass \( \gamma' \) of \( \gamma \).

A collection of ordered pairs of sets \( P \) is called a pair basis if

\[
P = \{P = (P_1, P_2) | P_i \subseteq X\}
\]

such that
On quasi-semidevelopable spaces

(1) \( P_1 \subset P_2 \) and \( P_1 \) is open, and
(2) for every \( x \) and for every neighborhood \( U \) of \( x \) there exists a \( P \) in \( P \) such that
\[
x \in P_1 \subset P_2 \subset U.
\]

A \( T_1 \) space \( X \) is said to be an \( M_\infty \)-space if \( X \) has a \( \sigma \)-cushioned pair basis [5].
It is well known that \( M_1 \)-space \( \rightarrow M_\infty \)-space \( \rightarrow M_3 \)-space. For the qs-developable spaces with closure preserving property we have following theorem.

**Theorem 3.2.** A regular and closure preserving qs-developable space is an \( M_\infty \)-space.

**Proof.** Let \( \gamma = (\gamma_1, \gamma_2, \ldots) \) be a qs-development for \( X \). Since each \( \gamma_n \) is closure preserving
\( B_n = \{ St(x, \gamma_n) \mid x \in X \} \) is also closure preserving.

**Theorem 3.3.** A regular space \( X \) has a closure preserving semi-development if and only if \( X \) has a closure preserving qs-development.

**Proof.** The necessity is trivial. Let \( \gamma = (\gamma_1, \gamma_2, \ldots) \) be a closure preserving qs-development for \( X \). Then \( X \) is an \( M_\infty \)-space by theorem 3.2. Since \( X \) is also an \( M_\infty \)-space,
\( X - \gamma_n^* = \bigcup U_n \) where each \( U_n \) is open. Let \( \zeta_n = \gamma_n \cup \{ U_n \} \). Then for each \( n \) and each
\( k \), \( \zeta_n \) is clearly a covering of \( X \) and is closure preserving. We show that \( \zeta = \{ \zeta_n \mid n = 1, 2, \ldots \} \) is a semi-development for \( X \). For each \( x \) and each \( n, k \) \( St(x, \zeta_n) \) is a neighborhood of \( x \). Let \( U \) be an open set containing \( x \). There exists a number \( n \) such that \( x \in St(x, \gamma_n) \subset U \). Since \( x \in \gamma_n^* \), there exists a number \( k \) such that \( x \in U_n \).
This implies that \( x \in St(x, \gamma_n) = St(x, \zeta_n) \subset U \), and completes the proof.

**Example 3.4.** There is a regular closure preserving semi-developable (hence qs-developable by the Theorem 3.3) space which is not metrizable.
Let \( R \) be the real line and \( Q \) be the set of rational numbers. We also use the notation \( \langle x, y \rangle \) denoting the point \( (x, y) \in R \times R \) to distinguish it from \( (s, t) \) which is an open interval. For \( x \in R \) put
\[
L_x = \langle x, y \rangle \mid \langle x, y \rangle \in R \times R, \ 0 < y \rangle
\]
and
\[
X = R \cup \{ L_x \mid x \in R \}.
\]
Now we define a basis for \( X \) as follows: For \( s, t \in Q \) and \( z = \langle x, w \rangle \in L_x, 0 < s < w < t \),
we put
\[
U_{x, s, t}(z) = \langle x, y \rangle \mid s < y < t \}
\]
and \( A \) to be the set of all such \( U_{x, s, t}(z) \). For \( r, s, t \in Q \) and \( z \in R, s < z < t \) and \( r > 0 \), we put
\[
V_{r, s, t}(z) = (s, t) \cup (\bigcup \{ \langle w, y \rangle \mid 0 < y < r, \ w \in (s, t) - \{ z \} \}),
\]
and \( B \) to be the set of all such \( V_{r, s, t}(z) \).
Now let \( U = A \cup B \). Then it can be easily shown that \( U \) is a \( \sigma \)-closure preserving basis.
making $X$ to be a nonmetrizable first countable $M_t$-space. For $z \in R$, $s < z < t$ let
\[ W_{r,s,t}(z) = (s, t) \setminus \{z\} \cup \{(x, y) | 0 < y < r\} \]
and
\[ U_{s,t} = \{U_{s,t}(x) | s < w < t, x \in R \text{ and } z = \langle x, w \rangle\}, \]
\[ W_{r,s,t} = \{W_{r,s,t}(x) | s < z < t \text{ and } z \in R\}. \]
Then
\[ \{U_{s,t} | s, t \in Q\} \cup \{W_{r,s,t} | r, s, t \in Q\} \]
is a closure preserving qs-development for $X$.

Alexander introduced a class of cushioned pair semi-developable spaces [2] and proved that a space is metrizable if and only if it is $T_0$ and has a cushioned pair semi-development. We generalize the concept by defining a cushioned pair qs-development and show that such a space is $M_t$.

**DEFINITION 3.5.** A space is **cushioned pair qs-developable** if there exist two qs-developments $\gamma, \delta$ for the space such that

1. each $\gamma_x$ is cushioned in $\delta_x$ and
2. for each $x$ and each open set $U$ containing $x$ there exists a number $n$ such that $x \in St(x, \gamma_x) \subseteq St(x, \delta_x) \subseteq U$.

It is not so difficult to show that if $X$ is cushioned pair qs-developable in the sense of [8], that is, if
\[ \{\text{the set of isolated points}\} \subseteq \gamma_1 \subseteq \gamma_2 \subseteq \cdots, \]
then this implies the above definition. For if $x$ is not an isolated point and $U$ be an open set containing $x$, there is an $m$ such that $x \in St(x, \gamma_m) \subseteq U$. Since $X$ is $T_1$, there must exist an $n > m$ such that $x \in St(x, \delta_n) \subseteq U$.

For this $n$, we have $x \in St(x, \gamma_n) \subseteq St(x, \delta_n) \subseteq U$.

From the definition cushioned pair semi-development is a cushioned pair qs-development. A cushioned pair qs-developable $T_0$-space is regular.

Let $\gamma$ and $\delta$ be collections of subsets of a space. We define that $\gamma$ is **weakly cushioned** in $\delta$ if there exists a mapping $D : \gamma \to \delta$ such that

1. $C \subseteq D(C)$ for each $C$ in $\gamma$ and
2. for each subclass $\gamma' \subseteq r$
\[ \text{cl}(\cup_{c \in \gamma} C \cup \text{cl}(D(C))). \]

This is a slight generalization of the definition 3.1.

A space is defined to be **weakly cushioned pair qs-developable** if there exist two qs-development $\gamma$ and $\delta$ such that

1. each $\gamma_x$ is weakly cushioned in $\delta_x$ and
2. for each $x$ and each open set $U$ containing $x$ there is a number $n$ such that
On quasi-semidevelopable spaces

\[ x \in \text{St}(x, \gamma_n) \subseteq \text{St}(x, \delta_n) \subseteq U. \]

It is clear that if \( X \) is cushioned pair qs-developable, then it is weakly cushioned pair qs-developable. If a space \( X \) is regular and has a closure preserving qs-development \( \gamma = (\gamma_1, \gamma_2, \ldots) \), then it has a weakly cushioned pair qs-development since \( \gamma \) is weakly cushioned in itself and is closure preserving.

**Theorem 3.6.** If a regular space \( X \) has a weakly cushioned pair qs-development, then it is an \( M_3 \)-space.

**Proof.** Let \( \gamma \) is weakly cushioned in \( \delta \) under the mapping \( D \) and \( P_n \) be a collection of ordered pairs \( P = (P_1, P_2) \) such that \( P_1 = \text{Int} \left( \text{St}(x, \gamma_n) \right) \), \( P_2 = \bigcup \text{cl}(D(C)) \) where \( C \) is a member of \( \Gamma_n \) containing \( x \). Then \( P_n \) is cushioned. It is easy to show that \( \bigcup P_n \) is a pair basis. This completes the proof.

From the above theorem we have following corollary which is a generalization of the theorem 2, 3 of [8].

**Corollary 3.7.** A regular cushioned pair qs-developable space is Nagata (a first countable stratifiable) space.

**References**


Seoul National University